

# Oligopolistic Markets with Sequential Search and Asymmetric Information\*

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## Abstract

A large variety of markets, such as retail markets for gasoline or mortgage markets, are characterized by firms offering a fairly homogenous good produced at virtually the same cost. The present paper provides a theoretical examination of this type of market by developing a sequential search model *with incomplete information* where consumers are uninformed about the underlying production cost. We characterize a perfect Bayesian equilibrium in which consumers follow reservation price strategies and provide a sufficient condition for such an equilibrium to exist. Firms strategically exploit consumers being uninformed about their production cost, and set on average higher prices compared to the standard complete information model. The importance of asymmetric information vanishes, however, when the number of firms becomes very large. Further, we find that expected prices and consumer welfare might be non-monotonic in the number of firms.

**JEL Classification:** D40; D83; L13

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# 1 Introduction

Consider a consumer who observes the price of gasoline at a gas station. Knowing that prices between different stations may vary considerably, the consumer must decide whether to buy at the observed price or search for a better deal elsewhere. When making this decision, she must estimate how much of the observed price is due to common factors affecting all gasoline stations in a similar way, e.g. the price of crude oil, and how much is due to idiosyncratic factors affecting the particular seller being visited. If the consumer believes that common factors are more relevant in determining the price, she might consider searching for a cheaper gas station not worthwhile and hence buy at the observed price. Conversely, if she believes that the station charges a particularly high price compared to other stations, she will probably find it optimal to look for a better deal. A key feature of this problem is that the consumer must take her decision under *incomplete information*: she is uncertain about the gas station's input (production) cost. Moreover, information is *asymmetric*, since gasoline retailers are obviously aware of this cost.<sup>1</sup>

In this paper we study how asymmetric information between firms and consumers affects equilibrium in a market like the one described above. To this end, we introduce this feature into the standard sequential search model with homogeneous goods, as developed by Stahl (1989). In our model, finitely many firms sell a homogenous product on an oligopolistic market, with all firms facing the same stochastic production cost and being aware of its realization. Consumers have inelastic demand and engage in sequential search for low prices. Unlike Stahl's sequential search model and most search literature, consumers do not observe the firms' production cost realization. Instead, they hold prior beliefs about the distribution of production costs and update these beliefs as they observe prices.<sup>2</sup>

In this environment we examine the properties of a *perfect Bayesian equilibrium satisfying a reservation prices property* (PBERP). In such an equilibrium, firms use mixed strategies and sample their prices from an optimal distribution that has no mass points; consumers employ a (non-stationary) reservation price rule, i.e., observing a certain price the consumer buys if this price is below her current reservation price, and searches for a lower price otherwise. At the relevant reservation price the consumer is indifferent between buying and searching for a better deal. Notice that a consumer who observes a price in the first round and continues to

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<sup>1</sup>This feature is not only found in gasoline markets, but characterizes many environments such as insurance or mortgage markets.

<sup>2</sup>In our working paper, Janssen, Pichler, and Weidenholzer (2009), we provide empirical evidence for the model under investigation. In particular, we show that the price dispersion observed in the gasoline retail market in Vienna, Austria, is consistent with a model of consumer search and difficult to reconcile with a model of horizontal product differentiation.

search will update her beliefs about the firms' underlying production cost on the basis of her first price observation, and therefore she will (generically) have a different reservation price in the second search round. A consequence of incomplete information within a sequential search framework thus is that, if there exists an equilibrium where optimal search behavior in each round is characterized by a reservation price, the reservation price must depend on the history of price observations. This history-dependence renders the analysis considerably more complicated compared to the complete information case, which may have led the literature to restrict attention to other, often less plausible, search protocols when studying information asymmetry in consumer search models.<sup>3</sup> Our paper attempts to fill this gap in the literature by examining a *sequential* search framework with asymmetric information. Within this framework, we focus on three questions. What are the characteristics of a PBERP? Under which conditions does such an equilibrium exist? And what is the role of asymmetric information within a sequential search framework?

Concerning the first question, we first show that in a PBERP no firm will set a price larger than the consumers' first round reservation price. To gain intuition for this finding, assume a firm would charge the upper bound, and that this upper bound is higher than the consumers' first-round reservation price. This firm would obviously not have any sales in the first search round; moreover, in later search rounds potential customers visiting the firm already have lower price observations in their pockets<sup>4</sup>, and thus the firm would not have any sales in later search rounds either. This behavior cannot be optimal. The central step in characterizing equilibrium thus turns out to be the characterization of the consumers' first-round reservation price. This is not a straightforward task, however, because consumers may in principle search more than once, leading to multiple instances of updating of their production cost beliefs. However, as we show in this paper, this cannot be the case in a PBERP. Importantly, a consumer who observes the first-round reservation price in a PBERP and, being indifferent, continues to search has a strictly higher reservation price in round two than in round one. This implies that the first-round reservation price is the price at which a consumer is indifferent between buying now and continuing to search exactly one more round, similar to the complete information model.

Concerning our second research question, we show that existence of a PBERP is not trivially

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<sup>3</sup>For example, Dana (1994) considers a model of asymmetric information with *newspaper* search, i.e., uninformed consumers can learn all prices at once by paying a search cost. He emphasizes, however, that the "... assumption of newspaper search is clearly restrictive and is not as realistic as the assumption that search is sequential (...). However, analyzing sequential search equilibrium under incomplete information is extremely difficult since consumers could in principle search more than once and hence more than one instance of Bayesian updating could occur." (p. 747).

<sup>4</sup>Recall that the price distribution in a PBERP has no mass points.

guaranteed. This finding is closely related to Rothschild (1974), who shows that an optimal consumer search strategy does not need to satisfy a reservation prices property if the price distribution is unknown to the consumer, and thus obtaining price observations also has an informational value as consumers update their beliefs about the true price distribution from which prices are sampled. In an environment where the price distribution is exogenously given, Rothschild shows that the consumer search strategy satisfies a reservation price property if for any two prices the difference between these two prices is smaller than the difference in their informational content. Using similar ideas in an environment where the price distribution is endogenously determined, and using specific properties the equilibrium price distribution has to satisfy, we specify a sufficient condition for the existence of a PBERP. In particular, we show that a PBERP exists in markets with either (i) a small support of the production cost distribution, (ii) relatively large search costs, (iii) relatively many firms, or (iv) relatively few shoppers. If a PBERP exists, we show that it is necessarily unique.

We extend Rothschild's logic in our framework by arguing that our condition is also "necessary" in the sense that, if the condition fails to hold, then one can always find distributions of the production cost such that a PBERP does not exist. Moreover, we provide an example showing that even when the production cost is uniformly distributed, a reservation price equilibrium fails to exist if our condition is not satisfied. Thus, incomplete information introduces significant changes to the sequential search model with respect to the existence of reservation price equilibria.

Concerning our third question, we have the following observations. First, in the complete information benchmark (which is essentially the Stahl model with unit demand), we show that the expected minimum price in the market is independent of the number of firms, while the expected price is increasing. Interestingly, when consumers are uncertain about the underlying cost level, the expected price and total consumer surplus over informed and uninformed consumers can be non-monotonic in the number of firms. Second, examining equilibrium price strategies used by firms, we show that the lower bound of the price distribution is increasing in the cost level while its upper bound is independent of the cost level.<sup>5</sup> Thus, the extent of equilibrium price dispersion under incomplete information decreases as the cost level rises. This constitutes an important and interesting difference to the complete information setting where the extent of price dispersion is independent of the production cost realization. Third, studying the welfare effects of incomplete information, we show that, from an ex-ante perspective,

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<sup>5</sup>In a PBERP, the upper bound must be equal to the first round reservation price of consumers. Since the latter cannot depend of the cost realization which is unknown to consumers, this property carries over to the upper bound.

consumer welfare is unambiguously lower under incomplete information and profits are unambiguously higher as compared to the environment with complete information. Both informed and uninformed consumers pay in expected terms higher prices. However, the ex-ante uncertainty concerning the price to be paid is typically much higher under complete information. Finally, when the number of firms in the industry becomes very large, the importance of asymmetric information vanishes and the expected price and the expected minimum price in the two models converges to each other. The main reason is that, with many firms, prices close to the reservation price do not convey information to consumers about the underlying cost of firms.

This paper contributes to a large and growing literature on equilibrium consumer search models starting from seminal contributions by Reinganum (1979), Varian (1980), Burdett and Judd (1983), and Stahl (1989). In terms of the research question being addressed, the papers most closely related to our paper are Benabou and Gertner (1993) and Dana (1994).<sup>6</sup> Both papers consider, however, simplified search protocols and do not consider a sequential search protocol. Benabou and Gertner analyze a duopoly market where half the consumers observe one price and the other half observes the other price at no cost. The only decision consumers have to make is whether to also observe the price of the firm they have not yet observed at a search cost. Dana considers a model with two types of consumers (informed and uninformed) where the uninformed consumers are engaged in *newspaper search*. These consumers get a first price quote for free and, on the basis of this price, they decide whether or not to become fully informed about all prices by paying a search cost. Papers by Fershtman and Fishman (1992) and Fishman (1996) and the recent contributions by Yang and Ye (2008) and Tappata (2008) use frameworks similar to Dana (1994), but extend them to a dynamic setting. In such environments, these papers study asymmetric price adjustment to cost shocks, the so-called *rockets-and-feathers* pattern. To the best of our knowledge, our paper is the first to introduce incomplete information into a *sequential* consumer search model.

In a broader sense the current paper is related to recent work which elaborates on the role of information gathering and information processing in consumer search.<sup>7</sup> This literature focuses on obfuscation (Ellison and Wolitzky (2009), Ellison and Ellison (2009)), boundedly rational agents (see, e.g., Spiegler (2006)), or information gatekeepers on the internet (see, e.g., Baye and Morgan (2001)). Another strand of the literature makes progress on the policy implications of the consumer search literature on consumer protection policies (see, e.g., Armstrong, Vick-

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<sup>6</sup>Earlier work by Diamond (1971) and Rothschild (1974) has analyzed optimal search behavior in a world where the price distribution is unknown, but exogenously given. Recently, Gershkov and Moldovanu (2009) uncover some formal relations between optimal stopping rules in the consumer search literature and the problem of ensuring monotone allocation rules in dynamic allocation problems.

<sup>7</sup>Baye, Morgan, and Scholten (2006) surveys a wide range of consumer search models .

ers, and Zhou (2009)) or on the empirical implementation of consumer search models (see, e.g., Lach (2007), Hortaçsu and Syverson (2004) and Moraga-González and Wildenbeest (2008)).

The remainder of this paper is organized in the following way. In Section 2 we briefly discuss the standard sequential search model with *completely* informed consumers, establishing a theoretical benchmark for comparison of our incomplete information model. This section also contains some new results (on the expected price of fully informed consumers) and some new characterizations making it easier to work with the model. In Section 3 we develop our model with *incompletely* informed consumers, define a perfect Bayesian equilibrium and characterize its properties. In Section 4 we summarize the effects of incomplete information on consumer and producer welfare and compare the models when the number of firms becomes large. Finally, in Section 5 we conclude and discuss directions for future research. Proofs are provided in the Appendix.

## 2 Sequential search with completely informed consumers

We start our analysis by describing a sequential search model with *completely* informed consumers along the lines of Stahl (1989). This model will, at a later stage, serve as our benchmark to assess the implications of incomplete (asymmetric) information within the sequential search framework. Essentially, we modify Stahl's model along only two dimensions. First, to simplify the analysis we consider a model of inelastic demand as in Janssen, Moraga-Gonzalez, and Wildenbeest (2005) and, second, do not normalize marginal costs to zero. Solving the model for positive marginal costs is inevitable for our purposes, because later on we want to analyze and compare situations under different marginal cost levels.

### 2.1 Model

We consider an oligopolistic market where  $N \geq 2$  firms sell a homogenous good and compete in prices. Each firm  $n \in \{1, \dots, N\}$  faces the same production technology and the same marginal production cost, denoted by  $c$ . Without loss of generality, we normalize fixed costs to zero. Each firm's objective is to maximize profits, taking the prices charged by other firms and the consumers' behavior as given.

On the demand side of the market we have a continuum of risk-neutral consumers with identical preferences. Each consumer  $j \in [0, 1]$  has inelastic demand normalized to one unit, and holds the same constant evaluation  $v > 0$  for the good. Observing a price below  $v$ , consumers will thus either buy one unit of the good or search for a lower price. In the latter case, they have

to pay a search cost  $s$  to obtain one additional price quote, i.e. search is *sequential*. A fraction  $\lambda \in [0, 1]$  of consumers, the *shoppers*, have zero search cost. These consumers sample all prices and buy at the lowest price. The remaining fraction of  $1 - \lambda$  consumers – the *non-shoppers* – have positive search costs  $s > 0$ . These consumers face a non-trivial problem when searching for low prices, as they have to trade off the search cost with the (expected) benefit from search. Consumers can always come back to previously visited firms incurring no additional cost, i.e. we are considering a model of *costless recall*.<sup>8</sup> We assume that  $v$  is large relative to  $c$  and  $s$  so that  $v$  is not binding. In this section consumers are informed about the cost realization  $c$ .

## 2.2 Equilibrium

In this model, there exists a unique symmetric Nash equilibrium where consumer behavior satisfies a *reservation price property*. Moreover, Kohn and Shavell (1974) and Stahl (1989) argue that the reservation prices are stationary. That is, the consumers' reservation prices are independent from the history of price observations and the number of firms left to be sampled (provided there is still at least one firm left) and can therefore simply be denoted by  $\rho^k(c)$ .<sup>9</sup>

To characterize this equilibrium it is useful to introduce some more notation: we denote, for a given production cost  $c$ , the distribution of prices charged by firms by  $F^k(p|c)$ , its density by  $f^k(p|c)$ , and the lower- and upper- bound of its support by  $\underline{p}^k(c)$  and by  $\bar{p}^k(c)$ , respectively.

It is well-known that the presence of both shoppers and non-shoppers,  $\lambda \in (0, 1)$ , implies that there does not exist an equilibrium in pure strategies and that there are no mass points in the equilibrium price distribution. The main reason behind this observation is that firms face a tradeoff between setting low prices to cater to the shoppers and setting high prices to extract profits from the non-shoppers. Also, the upper bound of the equilibrium price distribution must satisfy  $\bar{p}^k(c) = \rho^k(c)$ , i.e., in a symmetric equilibrium no firm will set a price higher than the reservation price  $\rho^k(c)$ . Given these two observations, the equilibrium price distribution can be characterized by

**Proposition 2.1** *For  $\lambda \in (0, 1)$ , the equilibrium price distribution for the cost realization  $c$  is given by*

$$F^k(p|c) = 1 - \left( \frac{1 - \lambda}{\lambda N} \frac{\bar{p}^k(c) - p}{p - c} \right)^{\frac{1}{N-1}} \quad (1)$$

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<sup>8</sup>Janssen and Parakhonyak (2007) analyze the case where this assumption is replaced by costly recall.

<sup>9</sup>We use the superscript  $k$  to indicate variables and parameters of the model with completely informed consumers who know the production cost realization  $c$ .

respectively

$$f^k(p|c) = \frac{1}{N-1} \frac{\bar{p}^k(c) - c}{(p-c)^2} \left( \frac{1-\lambda}{\lambda N} \right)^{\frac{1}{N-1}} \left( \frac{\bar{p}^k(c) - p}{p-c} \right)^{\frac{2-N}{N-1}} \quad (2)$$

with support on  $[\underline{p}^k(c), \bar{p}^k(c)]$  with  $\underline{p}^k(c) = \frac{\lambda N}{\lambda N + 1 - \lambda} c + \frac{1-\lambda}{\lambda N + 1 - \lambda} \bar{p}^k(c)$  and  $\bar{p}^k(c) = \rho^k(c)$ .

The proof follows essentially Stahl (1989) and is therefore omitted.

Having characterized the Nash equilibrium price distribution conditional on the reservation price  $\rho^k(c)$ , we turn to optimal consumer behavior. Given a distribution of prices  $F^k(p|c)$  and an observed price  $p'$ , it is straightforward to argue that the reservation price  $\rho^k(c)$  is implicitly determined by

$$v - \rho^k(c) = v - s - \int_{\underline{p}^k(c)}^{\rho^k(c)} p f^k(p|c) dp.$$

Using the result that the equilibrium price distribution satisfies  $\bar{p}^k(c) = \rho^k(c)$ , this condition boils down to

$$\rho^k(c) = s + E^k(p|c). \quad (3)$$

As the next lemma shows, this relation between reservation price and expected price allows us to derive a simple formula for the expected price conditional on the cost realization  $c$ ,  $E^k(p|c)$ .

**Lemma 2.1** *The expected price conditional on the cost realization  $c$ ,  $E^k(p|c)$ , is given by*

$$E^k(p|c) = c + \frac{\alpha}{1-\alpha} s, \quad (4)$$

where  $\alpha = \int_0^1 \frac{1}{1 + \frac{\lambda N}{1-\lambda} z^{N-1}} dz \in [0, 1)$ .

Janssen and Moraga-González (2004) have shown that  $\alpha$  is increasing in  $N$ . It follows that the expected price is increasing in  $N$  as well.

Note that (3) and (4) imply the following simple expression for the reservation price,

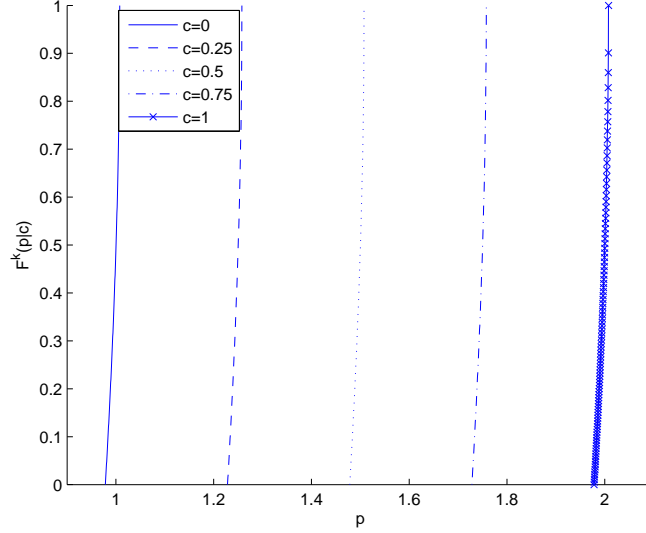
$$\rho^k(c) = \bar{p}^k(c) = c + \frac{s}{1-\alpha}. \quad (5)$$

The reservation price is thus a constant markup over the cost, with the size of the markup being determined by the model's parameters. Note further that, by Proposition 2.1,  $\underline{p}^k(c)$  is a weighted average of  $c$  and  $\rho^k(c)$ . Consequently, it immediately follows that, provided  $s > 0$ , the lower bound satisfies  $\underline{p}^k(c) > c$ . Thus firms make positive profits when charging prices according to  $F^k(p|c)$ . Furthermore, the following result obtains:

**Corollary 2.1** *The equilibrium price spread, i.e. the difference between the upper bound and the lower bound of the price distribution, is independent of the realized cost level  $c$  and given by*

$$\bar{p}^k(c) - \underline{p}^k(c) = \frac{\lambda N}{\lambda N + 1 - \lambda} \frac{s}{1-\alpha}. \quad (6)$$

Figure 1: Price distributions



Parameters:  $N = 3$ ,  $s = 0.01$ ,  $\lambda = 0.01$ ,  $c \sim U(0, 1)$

The proof follows from (5) and Proposition 2.1. What is interesting about Proposition 2.1 is that a change in  $c$  leads to a one to one shift in the price distribution, leaving the extent of price dispersion unaffected. In Figure 1 we highlight this point by plotting price distributions for various realizations of cost.

Note at this stage that, conditional on the cost  $c$ , the average price paid by a fraction  $1 - \lambda$  of consumers, i.e. the non-shoppers, is equal to  $E^k(p|c)$  as given in (4). This is, however, not the (average) price paid by the  $\lambda$  shoppers who observe all prices in the market and buy at the cheapest firm. This latter price is given by  $E^k(p_\ell|c)$ , with  $p_\ell = \min\{p_1, p_2, \dots, p_N\}$ . As firms choose prices randomly and independent from each other, it follows that the distribution of  $p_\ell$  is given by

$$F_\ell^k(p_\ell|c) = 1 - [1 - F^k(p|c)]^N. \quad (7)$$

Stahl (1989), and recently Morgan, Orzen, and Sefton (2006), and Waldeck (2008), show that in their models the shoppers observing all prices are better off with entry. Surprisingly, it turns out that with unit demand and an endogenous reservation price the expected minimum price is independent of  $N$ :

**Proposition 2.2** *The expected minimum-price conditional on the cost realization  $c$ ,  $E^k(p_\ell|c)$ , is given by*

$$E^k(p_\ell|c) = c + \frac{1 - \lambda}{\lambda} s, \quad (8)$$

and therefore independent on  $N$ .

In our model, entry has three different effects on the expected minimum price. First, the reservation price is increasing in the number of firms. As this implies that prices tend to be higher with more firms, this effect unambiguously makes shoppers worse off. A second effect is that shoppers sample more firms when there are more firms available in the market. This has the unambiguous effect that for a given price distribution shoppers are better off. Finally, price dispersion increases with more firms in the industry, implying that firms concentrate more on lower and higher prices at the expense of moderate prices. In fact, the lower bound of the price distribution decreases in the number of firms. Surprisingly, the Proposition shows that the net result of these three effects is zero. The difference with earlier results can be explained as follows. Stahl (1989) considers a model with downward sloping (instead of unit) demand, while Morgan, Orzen, and Sefton (2006) consider Varian's (1980) model where the consumer reservation price is exogenously given by the willingness to pay and consumers have unit demand.<sup>10</sup> The two papers together may suggest that when we consider the case of unit demand and the reservation price of non-shoppers being endogenously determined, the shoppers are also better off with entry. Our result shows that this is not the case.

Finally, we compute the unconditionally expected prices  $E^k(p) = \int E^k(p|c)dc$  and  $E^k(p_\ell) = \int E^k(p_\ell|c)dc$ . These expected prices will later on be important to assess consumer welfare in the economy, as  $v - E^k(p_\ell)$  is the expected equilibrium consumer surplus attained by shoppers whereas  $v - E^k(p)$  is expected equilibrium surplus of the  $1 - \lambda$  non-shoppers.<sup>11</sup> Formally, we obtain:

**Corollary 2.2** *The unconditionally expected prices  $E^k(p)$  and  $E^k(p_\ell)$  are given by*

$$E^k(p) = E(c) + \frac{\alpha}{1 - \alpha}s, \quad \text{and} \quad (9)$$

$$E^k(p_\ell) = E(c) + \frac{1 - \lambda}{\lambda}s, \quad (10)$$

where  $E(c) = \int_{\underline{c}}^{\bar{c}} cg(c)dc$ .

The proof follows trivially from Lemma 2.1 and Proposition 2.2. Note that both ex-ante expected prices take the form of a markup over the ex-ante expected cost, with the size of the respective markup being determined by the model's parameters, where it is interesting to note that the mark-up for the expected minimum price is independent of  $N$ , and the mark-up of the expected price is increasing in  $N$ .

<sup>10</sup>Strictly speaking, Stahl (1989) only proves this result for the case of infinitely many firms. For a finite number of firms he provides some numerical results.

<sup>11</sup>We will compute expected consumer surplus in the very beginning of the game, i.e. before the cost level  $c$  is drawn from  $g(c)$ .

### 3 Sequential search with incompletely informed consumers

We now turn to the analysis of the incomplete information model and modify the model presented in Section 2 by postulating that consumers are uninformed about the firms' production cost. Let nature randomly draw  $c$  from a continuous distribution  $g(c)$  with compact support on  $[\underline{c}, \bar{c}]$ . Consumers do not know the cost realization and they all hold correct prior beliefs about the production cost distribution and update their beliefs according to Bayes' rule as they observe prices.

#### 3.1 On out-of-equilibrium beliefs

In the model with incompletely informed consumers, the exact specification of out-of-equilibrium beliefs plays an important role in determining reservation prices. To see this point, assume that consumers hold out-of-equilibrium beliefs that are such that, if a price above their reservation price is observed, they think that the lowest cost level has been realized with probability one and therefore continue to search. In such a case, in equilibrium no firm would set a price above the reservation price (more details will be given shortly) and therefore such a price observation is clearly an out-of-equilibrium event. Note that under these particular beliefs, one can support a (first round) "reservation price" with the property that a consumer who observes it will strictly prefer to buy instead of actually being indifferent between buying and searching for a lower price. In a complete information setting, this could never be a reservation price as consumers would then also be willing to buy at a slightly higher price. However, in the incomplete information case under these particular out-of-equilibrium beliefs, a consumer would prefer to search for lower prices thinking that the lowest cost level has been realized and thus that prices should be low. Consequently, there would be a discontinuity in the willingness of consumers to buy around this "reservation price".

However, we think that such a discontinuity is difficult to defend in a consumer search model and we certainly do not want the comparison between the complete and incomplete information settings to depend on the arbitrary choice of out-of-equilibrium beliefs. We therefore insist that, if at a reservation price consumers strictly prefer to buy, out-of-equilibrium beliefs should be such that consumers also should buy at a slightly higher price. This effectively defines the first round reservation price as the price at which the consumer is indifferent between buying and continuing to search, in a way similar to the familiar complete information search model. In the following, we limit attention to equilibria satisfying such a reservation prices property.

### 3.2 Equilibria with reservation prices property

We start by providing a formal definition of what we mean by *equilibria satisfying a reservation prices property*. This requires first to introduce some more notation. In particular, we denote by  $\rho_t(p_1, \dots, p_{t-1})$  the reservation price of a consumer in search round  $t$  who has observed prices  $p_1, \dots, p_{t-1}$  in the  $t - 1$  previous search rounds. Note that unlike the complete information model, any reservation price  $\rho_t(p_1, \dots, p_{t-1})$  held by consumers has to be independent of the production cost and that the reservation price  $\rho_1$  in the first round is not conditional on any price observation, and we write  $\rho_1 = \rho$ . We have: <sup>12</sup>

**Definition 3.1** *A perfect Bayesian equilibrium satisfying a reservation prices property (PBERP) is characterized by:*

- 1) *each firm  $n \in \{1, \dots, N\}$  uses a price strategy that maximizes its (expected) profit, given the competing firms' price strategies and the search behavior of consumers;*
- 2) *given the (possibly degenerate) distribution of firms' prices, consumers search optimally given their beliefs, and update their beliefs given their price observations (if possible) and formulate out-of-equilibrium beliefs whenever they observe a non-equilibrium price; moreover, optimal consumer search is of the following form:*
  - i) *after observing  $p_t = \rho_t(p_1, \dots, p_{t-1})$  in round  $t$  and  $p_1, \dots, p_{t-1}$  in previous rounds, the consumer is indifferent between buying and continuing to search;*
  - ii) *after observing any  $p_t < \rho_t(p_1, \dots, p_{t-1})$  in round  $t$  and  $p_1, \dots, p_{t-1}$  in previous rounds, the consumer buys.*

In what follows, we concentrate on the characterization of this type of equilibrium and determine conditions for existence.

### 3.3 Properties of PBERP

We first examine the properties of a PBERP, assuming that such an equilibrium exists. In the next subsection, we consider the existence question. A first observation is that in a PBERP, the upper bound of the price distribution has to be equal to the reservation price of consumers in the first search round, i.e.  $\bar{p}(c) = \bar{p} = \rho$  for all  $c \in [\underline{c}, \bar{c}]$ . Suppose this was not the case and that for some  $c$ ,  $\bar{p}(c) > \rho$ . If a firm charges  $\bar{p}(c)$ , it will not sell to shoppers in any PBERP, as  $\bar{p}(c)$

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<sup>12</sup>This definition is an adaptation of the reservation price equilibrium defined by Dana (1994) to the case of sequential search.

does not have positive probability and therefore shoppers observe lower prices with probability one. Furthermore, a firm setting  $\bar{p}(c)$  will not sell to non-shoppers either, as these consumers will continue to search after observing  $\bar{p}$  in the first search round, and will then find a lower price in a subsequent search round with probability one. On the other hand, it can also not be the case that for some  $c$ ,  $\bar{p}(c) < \rho$  since firms could profitably deviate to a price equal to  $\rho$  because non-shoppers would continue to buy.

Under asymmetric information firms face virtually the same maximization problem as in the complete information benchmark. The only major difference is that upper bound of the price distribution is now constant at  $\bar{p} = \rho$  for all realizations of the cost  $c$ . Formally:

**Proposition 3.1** *In any PBERP the equilibrium price distribution for the cost realization  $c$  is given by*

$$F(p|c) = 1 - \left( \frac{1 - \lambda}{\lambda N} \frac{\bar{p} - p}{p - c} \right)^{\frac{1}{N-1}} \quad (11)$$

respectively

$$f(p|c) = \frac{1}{N-1} \frac{\bar{p} - c}{(p - c)^2} \left( \frac{1 - \lambda}{\lambda N} \right)^{\frac{1}{N-1}} \left( \frac{\bar{p} - p}{p - c} \right)^{\frac{2-N}{N-1}} \quad (12)$$

with support on  $[\underline{p}(c), \bar{p}]$  with  $\underline{p}(c) = \frac{\lambda N}{\lambda N + 1 - \lambda} c + \frac{1 - \lambda}{\lambda N + 1 - \lambda} \bar{p}$  and  $\bar{p} = \rho$ .

The proof is omitted, as it is a straightforward extension of the proof of Proposition 2.1. Inspection of (11) reveals that  $F(p|c)$  first order stochastically dominates (FOSD)  $F(p|c')$  whenever  $c > c'$ . Furthermore, we have that  $\underline{p}(c)$  is increasing in  $c$ , implying that consumers who observe prices below  $\underline{p}(\bar{c})$  can rule out certain (high) cost realizations. Employing the techniques used in 2.1 we can write  $E(p|c)$  as

$$E(p|c) = (1 - \alpha)c + \alpha \bar{p}. \quad (13)$$

Figure 2 visualizes all these observations by plotting the price distributions  $F(p|c)$  for different realizations of the production cost  $c$ . This figure points to the fact that the price spread is decreasing in  $c$  as stated in the following corollary:

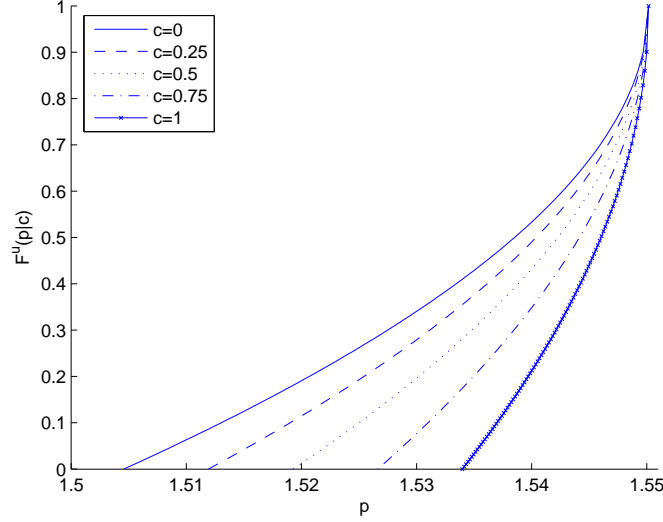
**Corollary 3.1** *If consumers are uninformed about the firms' cost realization, the price spread in a PBERP is equal to*

$$\bar{p} - \underline{p}(c) = \frac{\lambda N}{\lambda N + 1 - \lambda} (\bar{p} - c)$$

and therefore is decreasing in the cost level  $c$ .

Corollary 3.1 establishes a sharp contrast to the complete information model in which the price spread is independent of the realized cost level.

Figure 2: Price distributions



Parameters:  $N = 3$ ,  $s = 0.01$ ,  $\lambda = 0.01$ ,  $c \sim U(0, 1)$

Let us now turn the focus on consumers' search behavior. Assuming consumers have correct prior beliefs about the production cost and use Bayesian updating after observing a price  $p$ , let  $\delta(c|p)$  be the (posterior) probability density function of the production cost  $c$  conditional on a price observation  $p$ . By Bayes rule, we have that:

$$\delta(c|p) = \frac{g(c)f(p|c)}{\int_{\underline{c}}^{\bar{c}} g(c')f(p|c')dc'}. \quad (14)$$

It remains to specify consumers' out-of-equilibrium beliefs, i.e. beliefs on the cost level for price observations not in the support of the equilibrium price distribution. As argued above, we want to avoid out-of-equilibrium beliefs that create a discontinuity in the willingness of consumers to buy around the reservation price. To this end, we assume that for a price observation above the upper bound of the price distribution consumers hold the same beliefs on the cost level as if they had observed the upper bound, i.e.  $\delta(c|p) = \delta(c|\bar{p})$  for  $p > \bar{p}$ .<sup>13</sup>

In the following, we derive several lemmas that will prove useful to examine the properties of PBERP. First, we identify an important feature of Bayesian updating in our framework that plays a key role in our main results: a consumer who has observed a price  $p \in [\underline{p}(\bar{c}), \bar{p}]$  will put more probability mass on higher realization of the production cost and less mass on lower realizations of the production cost than under the prior distribution  $g(c)$ .

<sup>13</sup>As at prices below the lower bound of the price distribution in the lowest cost scenario consumers buy regardless of their beliefs on the realized cost level, beliefs in these states are irrelevant.

**Lemma 3.1** For any  $p \in [\underline{p}(\bar{c}), \bar{p}]$ , the posterior distribution of cost levels  $\delta(c|p)$  first order stochastically dominates the prior distribution  $g(c)$ .

Lemma 3.1 implies that non-shoppers who have observed any  $p \in [\underline{p}(\bar{c}), \bar{p}]$  expect a higher cost level  $E(c|p)$  than if they hadn't observed any price, i.e.

$$E(c|p) = \int_{\underline{c}}^{\bar{c}} \delta(c|p)cdc > \int_{\underline{c}}^{\bar{c}} g(c)cdc = E(c).$$

Moreover, we find that the higher the price observed in the interval  $p \in [\underline{p}(\bar{c}), \bar{p}]$ , the more optimistic the consumer is about the possibility of finding low prices if she continues searching. Formally,

**Lemma 3.2** For all  $p \in [\underline{p}(\bar{c}), \bar{p})$ , there is a unique cost level  $\hat{c}$  such that

$$\frac{\partial \delta(c|p)}{\partial p} \begin{cases} > 0 & \text{if } c < \hat{c} \\ = 0 & \text{if } c = \hat{c} \\ < 0 & \text{if } c > \hat{c}. \end{cases}$$

Consequently, for all  $p, p' \in [\underline{p}(\bar{c}), \bar{p}]$  with  $p' > p$ , the posterior distribution of cost levels  $\delta(c|p)$  FOSD  $\delta(c|p')$ .

Lemma 3.2 appears puzzling at first sight. However, the intuition behind it is readily seen: in the interval  $[\underline{p}(\bar{c}), \bar{p}]$ , the ratio of densities  $f(p|c')/f(p|c)$  is increasing in  $p$  for any pair  $c, c'$  with  $c' < c$ . This implies that higher prices in  $[\underline{p}(\bar{c}), \bar{p}]$  are *relatively* more likely under low costs than under high costs, which in turn explains why higher price observations in  $p \in [\underline{p}(\bar{c}), \bar{p}]$  lead the consumer to become more optimistic about the cost realization.

We now move on to the characterization of reservation prices under *incomplete* information. Recall that in the *complete* information model, the reservation price is defined as the price at which the consumer is indifferent between buying now and continuing to search. In the present context, this would translate into defining the (first round) reservation price  $\rho$  by the indifference condition

$$v - \rho = v - s - E(p|\rho),$$

with  $E(p|\rho) = \int_{\underline{c}}^{\bar{c}} E(p|c)\delta(c|\rho)dc$ . The reservation price  $\rho$  would thus be implicitly given by

$$\rho = s + \int_{\underline{c}}^{\bar{c}} E(p|c)\delta(c|\rho)dc. \quad (15)$$

However, under *incomplete* information reservation prices are not stationary and do depend on the search history, due to Bayesian updating of beliefs. The arguments used above may

hence not be valid, and it is not obvious that the first round reservation price should satisfy (15). In what follows, we however prove that (15) still provides a proper characterization of the reservation price. The intuition is the following: if a consumer in search round one observes the round one reservation price and decides to continue searching, she will find a price strictly below her round *two* reservation price with probability one, and thus necessarily buys in round two. Thus, the reservation price in round one is the price at which the consumer is indifferent between buying and searching one more firm. Lemma 3.3 establishes this result.

**Lemma 3.3** *In any PBERP, after observing the upper bound of the price distribution in the first search round, a consumer's reservation price in the second search round satisfies  $\rho_2(\rho) > \rho$ .*

It follows that there does *not* exist a PBERP where consumers follow a stationary reservation price search rule. This is an important difference with the complete information model. Using equation (13) and  $\rho = \bar{p}$  gives us

$$\begin{aligned}\bar{p} &= s + \int_{\underline{c}}^{\bar{c}} \left( (1 - \alpha)c + \alpha\bar{p} \right) \delta(c|\bar{p}) dc \\ &= s + (1 - \alpha) \int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc + \alpha\bar{p} \int_{\underline{c}}^{\bar{c}} \delta(c|\bar{p}) dc.\end{aligned}$$

Since  $\int_{\underline{c}}^{\bar{c}} \delta(c|\bar{p}) dc = 1$  and  $\int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc = E(c|\bar{p})$ , we further have that the reservation price is implicitly defined by

$$\rho = \bar{p} = E(c|\bar{p}) + \frac{s}{1 - \alpha}. \quad (16)$$

Substituting (16) into (13), we arrive at the following result concerning the conditional expected price. The claim concerning the conditional expected minimum price is a little more difficult to arrive at, but essentially follows from the fact that

$$(1 - \lambda)(E(p|c) - c) + \lambda(E(p_\ell|c) - c) = (1 - \lambda)(\rho - c), \quad (17)$$

i.e., the sum of the expected industry profits over shoppers and non-shoppers has to be equal to  $N$  times the expected individual profit, which equals  $(1 - \lambda)(\rho - c)$ .

**Proposition 3.2** *In any PBERP the conditionally expected prices  $E(p|c)$  and  $E(p_\ell|c)$  in a PBERP are given by*

$$E(p|c) = c + \frac{\alpha}{1 - \alpha} s + \alpha [E(c|\bar{p}) - c], \quad (18)$$

$$E(p_\ell|c) = c + \frac{(1 - \lambda)}{\lambda} (s + (1 - \alpha) [E(c|\bar{p}) - c]). \quad (19)$$

From Proposition 3.2 it immediately follows that the unconditionally expected prices  $E(p)$  and  $E(p_\ell)$  are given by

$$E(p) = E(c) + \frac{\alpha}{1-\alpha}s + \alpha[E(c|\bar{p}) - E(c)], \quad (20)$$

$$E(p_\ell) = E(c) + \frac{(1-\lambda)}{\lambda}(s + (1-\alpha)[E(c|\bar{p}) - E(c)]). \quad (21)$$

### 3.4 Existence of PBERP

Having established some properties any PBERP should satisfy, we now move to the existence question. Note that we have so far implicitly assumed that non-shoppers would like to buy at all prices below  $\rho$ . While this is straightforward to establish in the framework with complete information, it is not obvious under incomplete information since consumers update their beliefs about the true cost as they observe prices. In particular, after observing a price  $p < \underline{p}(\bar{c})$  a consumer may suddenly think that the cost is very low and thus may decide to continue searching. Moreover, we need to verify that for all cost realizations firms find it optimal to set the prices implicitly specified above. In particular, we need that  $\underline{p}(c) > c$  for all values of  $c$ . Again, under asymmetric information this condition is not automatically satisfied as the reservation price (and thereby the upper bound of the price distribution) is independent of the cost realization.

The next Proposition establishes a sufficient condition for the existence of PBERP with non-stationary reservation prices. Rothschild (1974) already observed that if consumers sample from an unknown distribution, it may happen that they prefer to buy at high prices, whereas they continue to search (and do not buy) at lower prices. He also provides a sufficient condition under which it is optimal for consumer to actually follow a reservation price strategy. The sufficient condition he states is that the price difference between any two prices should be smaller than the difference in informational content of these prices. Rothschild focuses, however, on the consumer search problem for a given (but unknown) price distribution. We show that these considerations also arise in consumer search models where firms are strategically choosing prices. Importantly, our sufficient condition is in terms of the exogenous parameters of the model.

**Proposition 3.3** *If*

$$\bar{c} - \underline{c} \leq \frac{\lambda N}{\lambda N + 1 - \lambda} \left( \frac{s}{1 - \alpha} \right), \quad (22)$$

*then a unique PBERP exists.*

The proof is based on the following considerations. We first show that observing a price  $p$  with  $\underline{p}(\bar{c}) < p < \rho$ , an uninformed consumer prefers to buy instead of continuing to search and

buy in a later round. Note that at these prices, consumers assign positive density to any cost realization  $c$  and by Lemma 3.1 become more pessimistic about the possibility of finding lower prices when continuing to search. We then examine lower price observations  $p' < \underline{p}(\bar{c})$ , where consumers can rule out certain high cost realizations. For a PBERP to exist, consumers must still find it optimal to buy at such prices. This, in turn, requires that consumers do not infer from observing a price  $p' < \underline{p}(\bar{c})$  that the cost is low enough so that continued search pays off. To rule out this case, we exploit the idea that a consumer who finds it optimal to buy at a price  $p'$  if he *knows* the cost realization is  $\underline{c}$ , i.e.  $p' \leq \rho^k(\underline{c})$ , certainly has to find it optimal to buy in the unknown cost case at the same price. Consequently, by imposing  $\underline{p}(\bar{c}) \leq \rho^k(\underline{c})$  we can guarantee that a consumer will find it optimal to buy at all price observations smaller than the reservation price  $\rho$ . This condition translates into inequality (22) characterizing the existence of a PBERP. Furthermore, inequality (22) also is sufficient to ensure that firms will set prices as specified above, i.e.  $\underline{p}(c) > c$  holds for all  $c \in [\underline{c}, \bar{c}]$ .

It is interesting to see when the condition in Proposition 3.3 holds. Clearly, this is the case when the support of the cost distribution  $\bar{c} - \underline{c}$  is small or  $s$  is large. More interestingly, it is also the case when  $N$  is large enough (for any given values of the other parameters). To see this, note that both  $\frac{\lambda N}{\lambda N + 1 - \lambda}$  and  $\alpha$  approach one as  $N$  approaches infinity; the RHS of inequality (22) therefore approaches infinity as well. Finally, note that when  $\bar{c} \leq \underline{c} + Ns$ , a PBERP exists also for small values of  $\lambda$ . To arrive at this observation, we evaluate

$$\frac{\lambda N}{\lambda N + 1 - \lambda} \frac{1}{1 - \int_0^1 \frac{1}{1 + \frac{\lambda N}{1 - \lambda} z^{N-1}} dz}$$

when  $\lambda$  is close to zero. Applying l'Hopital's Rule, we get that in a neighborhood of  $\lambda = 0$

$$\frac{\frac{N}{(\lambda N + 1 - \lambda)^2}}{\int_0^1 \frac{N z^{N-1} / (1 - \lambda)^2}{\left(1 + \frac{\lambda N}{1 - \lambda} z^{N-1}\right)^2} dz} = \frac{N}{\int_0^1 N z^{N-1} dz} = \frac{1}{\int_0^1 z^{N-1} dz} = N.$$

For  $\lambda$  close to 0, the right hand side of our inequality is thus approximately equal to  $\underline{c} + Ns$ .

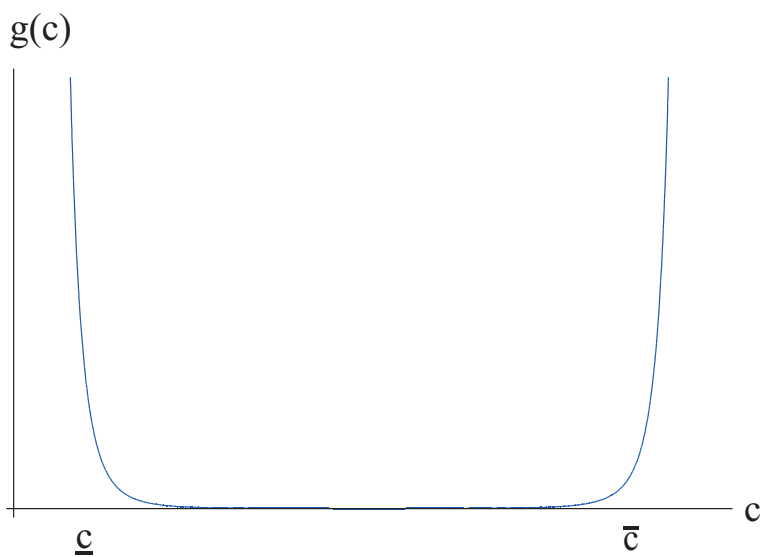
We summarize our findings regarding the existence of PBERP in the following corollary:

**Corollary 3.2** *A PBERP exists in environments with*

- (i) *a sufficiently small support of the cost distribution,  $\bar{c} - \underline{c}$ , and/or*
- (ii) *sufficiently large search costs  $s$ , and/or*
- (iii) *sufficiently many firms  $N$ , and/or*
- (iv) *a sufficiently small fraction of shoppers  $\lambda$ , provided that  $\bar{c} \leq \underline{c} + Ns$  holds.*

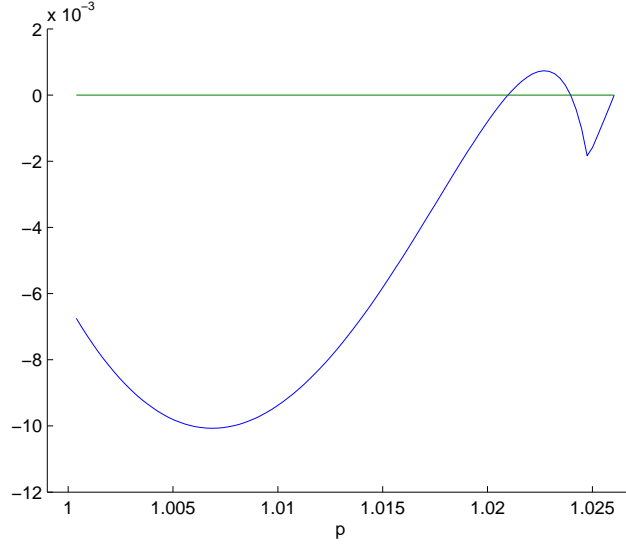
For general functions  $g(c)$  the condition established in Proposition 3.3 is “almost necessary” in the following sense. If  $\underline{p}(\bar{c}) > \rho^k(\underline{c})$ , then one can construct a density function of the cost parameter,  $g(c)$ , that is concentrated on values close to the two extremes  $\underline{c}$  and  $\bar{c}$  (see Figure 3) such that, after observing a price smaller than  $\underline{p}(\bar{c})$ , consumers suddenly consider it extremely likely that the cost is close to  $\underline{c}$ . In particular, if a price observation  $p$  is in the interval  $(\rho^k(\underline{c}), \underline{p}(\bar{c}))$  consumers will then prefer to search.

Figure 3: A cost distribution concentrated around the two extremes.



One may then wonder whether, if we restrict the prior cost distribution, existence of a PBERP may always be guaranteed. Considering a uniform distribution of production costs, the next example demonstrates that this is not the case. Figure 4 displays the net benefits of search in a duopoly market with search costs equal to  $s = 0.00675$ , a shopper-share equal to  $\lambda = 0.025$ , and production costs drawn from the uniform distribution  $U(0, 1)$ . As can easily be seen, for this parameter constellation no PBERP exists: the consumer does not prefer to buy for all prices below the potential reservation price defined by equation (16). While for prices between  $\rho = 1.0260$  and  $\underline{p}(\bar{c}) = 1.0248$  the consumer strictly prefers to buy, when observing prices slightly below  $\underline{p}(\bar{c})$ , the net benefits of search are increasing rapidly. Indeed, the net search benefits become positive for an interval of prices a bit below  $\underline{p}(\bar{c})$ . The reason is that when the consumer observes prices just below  $\underline{p}(\bar{c})$ , she infers that the expected production cost is relatively low and so is the expected price. When she would observe even lower prices, the search benefits increase rapidly and it becomes profitable not to buy at the observed price but to search for a lower price, even though the consumer would have bought had she observed

Figure 4: Net benefits of search



Parameters:  $N = 2$ ,  $s = 0.00675$ ,  $\lambda = 0.025$ ,  $c \sim U(0, 1)$

a slightly higher price.

In case a PBERP does not exist, it is important to know what type of equilibrium does exist. Unfortunately, it turns out this is a very difficult issue to resolve. For example, it can be shown that allowing for reservation prices which are more general than the ones in Definition 3.1 does not overcome the non-existence problem. Note that a more general reservation price strategy is a strategy according to which consumers buy if, and only if, they observe a price at or below a certain cut-off price, but where consumers are not indifferent between buying and continuing searching at the reservation price. As explained before, such reservation prices could be supported in our framework by specifying out-of-equilibrium beliefs in such a way that after observing prices higher than the upper bound of the price support (the reservation price), consumers believe that the underlying costs are very low and therefore strictly prefer to continue searching. The next result establishes that there are parameter values for which equilibria where consumers follow such strategies do not exist.

**Proposition 3.4** *If  $s$  is relatively small or  $\bar{c} - \underline{c}$  is relatively large and  $g(c)$  has a relatively high probability mass close to  $\underline{c}$ , then an equilibrium where consumers follow a reservation price strategy does not exist.*

Together with the obvious fact that there cannot be a hole in the prices at which the non-shoppers decide to buy with probability one, Proposition 3.4 implies that for some parameter

values consumers have to follow a mixed strategy in equilibrium. We leave it for further research to fully characterize these equilibria.

## 4 Welfare implications and the impact of entry

We are now ready to compare the two models more formally. The most important basis for this comparison is the examination of the split of welfare between uninformed consumers, informed consumers and firms generated in the two models and how this split of welfare depends on the number of firms in the industry. First, we assess the impact of incomplete information on (i) the ex-ante expected price, (ii) the ex-ante expected lowest price, and (iii) the ex-ante expected profit of firms by comparing the two complete and incomplete information scenarios.

**Proposition 4.1** *In the PBERP of the sequential search model with incomplete information,*

- *the ex-ante expected price paid by non-shoppers,  $E(p)$ ,*
- *the ex-ante expected price paid by shoppers,  $E(p_\ell)$ , and*
- *the ex-ante expected profit made by firms,*

*are higher than in the complete information model. Consequently, consumer surplus is lower and producer surplus is higher.*

Proposition 4.1 illustrates that consumer welfare is higher when the consumers are informed about the firms' production cost suggesting that policy interventions inducing observability of production cost benefit consumers. The main reason for this result is the following. As uninformed consumers update their beliefs about the underlying cost level upon observing a price, the reservation price in the unknown cost case is larger than the unconditional a priori expected cost (which is relevant in the known cost case), i.e., the expressions in equations (20) and (21) are larger than the ones in (9) and (10). It is, however, important to note that the ex ante variation of prices is much larger in the known cost case than in the unknown cost case, as a quick inspection of Figures 1 and 2 reveals. This is because the upper bound of the price distribution in the known cost case shifts with the cost parameter, while this is not the case in the unknown cost case. For the welfare comparison in our model, this is of no relevance as consumers are supposed to be risk neutral. However, in case consumers would be risk averse, it may be that the welfare comparison seriously depends on how consumers value the ex ante price variation. Further, we have that :

**Proposition 4.2** *The conditionally expected profits of firms are decreasing in the cost level  $c$  when consumers are uninformed about the cost realization, whereas these profits are independent of  $c$  when consumers are perfectly informed. In particular, conditionally expected profits*

*under incomplete information are higher (lower) for low (high) cost realizations compared to when consumers are perfectly informed.*

Again, the fact that in the known cost case the reservation price shifts with the underlying costs is responsible for this result.

Finally, we study the effect of changes in the number of firms on the welfare evaluation of the different groups in the industry, and especially how the differences between the two models depends on  $N$ . For the known cost case, these results are already summarized at the end of Section 2. It is difficult to evaluate the relevant expressions for the unknown cost case as it is difficult to evaluate the impact of  $N$  on the reservation price. We are able, however, to ascertain the following two results. First, when the number of firms in the industry is large, the difference between the two models becomes negligibly small, i.e., the complete information model provides a good approximation of the more complicated incomplete information model.<sup>14</sup> This is the content of the next Proposition.<sup>15</sup>

**Proposition 4.3** *If the number of firms increases without bound we have*

$$\begin{aligned}\lim_{N \rightarrow \infty} \left( \rho - E(\rho^k(c)) \right) &= 0 \\ \lim_{N \rightarrow \infty} \left( E(p) - E^k(p) \right) &= 0 \\ \lim_{N \rightarrow \infty} \left( E(p_\ell) - E^k(p_\ell) \right) &= 0.\end{aligned}$$

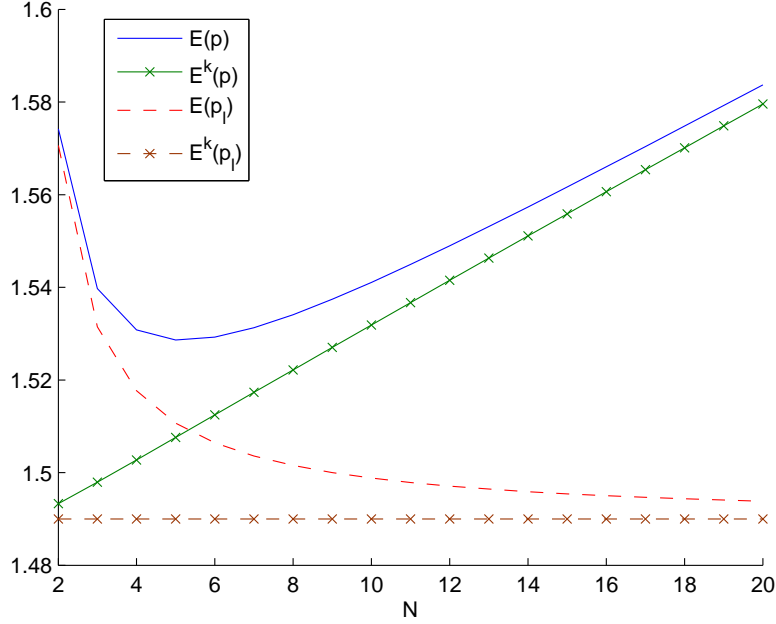
The proof of this proposition exploits the fact that around the upper bound of the price distribution (which is the reservation price), the price distributions in the unknown cost case are almost identical for different cost realizations. Bayesian updating yields therefore not more information and the reservation price is simply based on the ex ante expected cost. Thus, for large  $N$  the reservation price in the unknown cost case is very close to the ex ante expected reservation price in the known cost case. It is striking that the difference between the two models becomes negligible in competitive markets where  $N$  is large. This is in contrast to results in ? where, in a context where prices signal quality, the difference between the complete and incomplete information models remains for large  $N$ .

A second result is that the effects of increased competition in the unknown cost case may be non-monotonic. As can be verified from Figure 5, there are numerical examples where the expected price,  $E(p)$ , is first decreasing in the number of firms and starts to increase only above a certain number of firms in the market. The fact that the expected price  $E(p)$  may be

<sup>14</sup>We already know from Section 3.4 that the existence condition for PBERP is satisfied for large  $N$ .

<sup>15</sup>Note that we assume that the willingness to pay is large enough such that the constraint that prices must be below  $v$  is not binding.

Figure 5: The effects of entry on expected prices



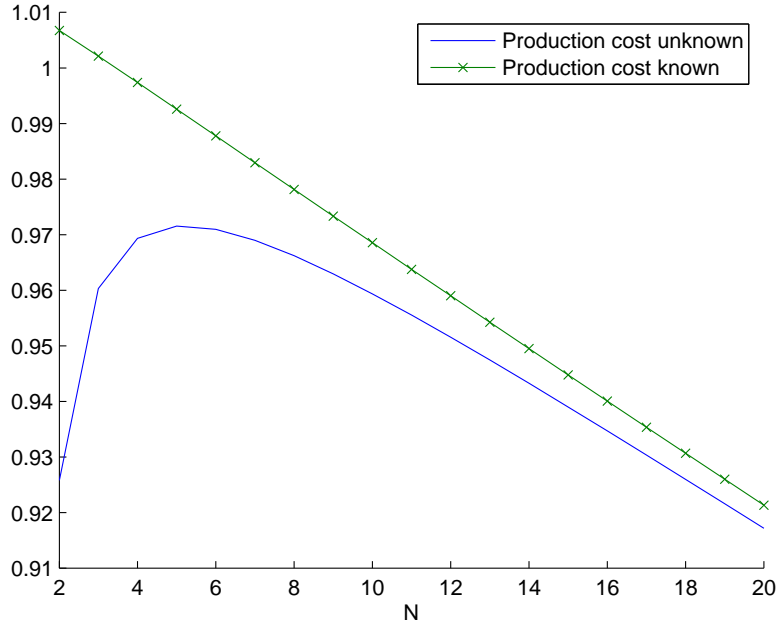
Parameters:  $s = 0.01$ ,  $\lambda = 0.01$ ,  $c \sim U(0, 1)$

decreasing in  $N$  for small  $N$  is in sharp contrast with the comparative static results in other search models in this direction, where expected prices are typically increasing in  $N$ . As  $\alpha$  is increasing in  $N$  it is clear from (20) that this result is due to the fact that  $\rho$  is decreasing in  $N$  as well. As the profits in our industry are given by  $(1 - \lambda)(\rho - c)$ , the industry's total profits reach a minimum in this example when there are five firms in the market. As consumer surplus is just the reverse of industry total profits in this model, total consumer surplus reaches a maximum for five firms as Figure 6 illustrates. This is in contrast to the known cost case (also depicted in Figure 6) where it can be easily verified that consumer welfare is decreasing in the number of firms. The non-monotonicity encountered in the unknown cost case points to the fact that it is extremely difficult to obtain analytic comparative static results for small  $N$ .

## 5 Conclusion

In this paper we have analyzed a sequential consumer search model with incomplete (asymmetric) information about the common underlying production cost of firms. We have characterized a perfect Bayesian equilibrium of this model satisfying a reservation prices property. In this equilibrium, firms sample prices from an optimal distribution and, in each search round, a con-

Figure 6: The effects of entry on consumer welfare



Parameters:  $s = 0.01$ ,  $\lambda = 0.01$ ,  $c \sim U(0, 1)$ ,  $v = 2$

sumer buys if she observes a price below her current reservation price and searches for a better deal otherwise. Unlike in the standard consumer search model, the reservation prices under incomplete information are not stationary but differ across search rounds. This is due to consumers updating their beliefs about the production cost level when observing prices. We have further shown that an equilibrium with the properties just outlined exists for a relevant range of parameter values, but there are cases where a reservation price equilibrium does not exist.

Comparing our environment to the complete information search model, we have shown that both the average price and the expected lowest price in the market are higher, and consumer welfare is thus lower, under incomplete information. This difference between the two model becomes, however, negligibly small when the number of firms in the industry is large. When the number of firms is relatively small, the unknown cost model also may give rise to a non-monotonic relationship between expected consumer welfare and the number of firms: where consumer surplus is first increasing in the number of firms and then decreasing so that from a consumer perspective, there is an optimal number of firms in the industry. We have furthermore demonstrated that the average profit margin charged by firms and the extent of equilibrium price dispersion are decreasing in the cost level, which is not the case in the complete information model.

There are several directions for future research. From a theoretical perspective, the present paper does not answer the question which type of equilibria do exist in case a PBERP does not exist. Moreover, from our welfare comparison it becomes clear that it is interesting to consider a search model where consumers are risk averse. From a more empirical perspective, it would be nice to confront our model with data from retail gasoline or other markets. One issue that needs to be addressed in such research the current paper could be extended to build a reasonable dynamic model of the retail gasoline market, where consumers beliefs about retailers' cost are endogenously determined.

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## Proofs

**Proof of Lemma 2.1.** The expected price  $E^k(p|c)$  is given by

$$E^k(p|c) = \int_{\underline{p}}^{\rho^k(c)} p f^k(p|c) dp = c + \int_0^1 (p - c) dF^k(p|c). \quad (23)$$

Introducing

$$z = 1 - F^k(p|c) = \left( \frac{1 - \lambda \rho^k(c) - c}{\lambda N (p - c)} \right)^{\frac{1}{N-1}},$$

we have that

$$p - c = (\rho^k(c) - c) \frac{1}{1 + \frac{\lambda N}{1-\lambda} z^{N-1}}.$$

This allows us to rewrite expression (23) as

$$E^k(p|c) = (1 - \alpha)c + \alpha \rho^k(c),$$

with  $\alpha = \int_0^1 \frac{1}{1 + \frac{\lambda N}{1-\lambda} z^{N-1}} dz \in [0, 1]$ . Substituting (3) allows us then to write the expected price as in the statement of the lemma. ■

### Proof of Proposition 2.2

Firms expect to make a profit of  $E^k(p|c) - c$  on the  $(1 - \lambda)$  non-shoppers and a profit of  $E^k(p_\ell|c) - c$  on the  $\lambda$  shoppers. Moreover, we know that in a mixed strategy equilibrium all prices in the support of  $F^k(p|c)$  have to yield the same profit and that this profit has to equal the profit made when charging the reservation price. Hence we have that

$$(1 - \lambda)(E^k(p|c) - c) + \lambda(E^k(p_\ell|c) - c) = (1 - \lambda)(\rho^k(c) - c) \quad (24)$$

Substituting the reservation price defined in (5) and the expected price, the expression for the expected minimum price follows as given in the statement of the Proposition. As  $c$ ,  $\lambda$  and  $s$  are exogenous parameters, it follows that  $E^k(p_\ell|c)$  is independent of  $N$ . ■

**Proof of Lemma 3.1** Recall that the probability density function of the production cost  $c$  conditional on a price observation  $p$  is given by

$$\delta(c|p) = \frac{g(c)f(p|c)}{\int_{\underline{c}}^{\bar{c}} g(c')f(p|c')dc'} = \frac{g(c)}{\int_{\underline{c}}^{\bar{c}} g(c')\frac{f(p|c')}{f(p|c)}dc'} \equiv \frac{g(c)}{y(c;p)}.$$

Note that if  $y(c;p)$  is monotonically decreasing in  $c$ , then  $\delta(c|p)$  first order stochastically dominates  $g(c)$ . In the following, we show that  $y(c;p)$  is indeed monotonically decreasing in  $c$  for

price observations in the interval  $[p(\bar{c}), \bar{p}]$ . To this end, note that for  $p \in [p(\bar{c}), \bar{p}]$  the function  $y(c; p)$  is given by

$$\begin{aligned} y(c; p) &= \int_{\underline{c}}^{\bar{c}} g(c') \frac{\bar{p} - c'}{\bar{p} - c} \left( \frac{p - c}{p - c'} \right)^{\frac{N}{N-1}} dc' \\ &= \int_{\underline{c}}^{\bar{c}} g(c') \frac{\bar{p} - c'}{(p - c')^{\frac{N}{N-1}}} \frac{(p - c)^{\frac{N}{N-1}}}{\bar{p} - c} dc'. \end{aligned}$$

Its derivative with respect to  $c$  is hence given by

$$\begin{aligned} \frac{\partial y(c; p)}{\partial c} &= \int_{\underline{c}}^{\bar{c}} g(c') \frac{\bar{p} - c'}{(p - c')^{\frac{N}{N-1}}} \left[ \frac{-\frac{N}{N-1} (p - c)^{\frac{N}{N-1}-1} (\bar{p} - c) + (p - c)^{\frac{N}{N-1}}}{(\bar{p} - c)^2} \right] dc' \\ &= \int_{\underline{c}}^{\bar{c}} g(c') \frac{\bar{p} - c'}{(p - c')^{\frac{N}{N-1}}} \frac{(p - c)^{\frac{N}{N-1}}}{(\bar{p} - c)^2} \left[ 1 - \frac{N}{N-1} \frac{(\bar{p} - c)}{(p - c)} \right] dc' < 0 \end{aligned}$$

as  $\left[ 1 - \frac{N}{N-1} \frac{(\bar{p} - c)}{(p - c)} \right] < 0$ . ■

**Proof of Lemma 3.2.** Taking the derivative of  $\delta(c|p)$  with respect to  $p$  yields

$$\frac{\partial \delta(c|p)}{\partial p} = - \frac{[\delta(c|p)]^2}{g(c)} \int_{\underline{c}}^{\bar{c}} g(c') \frac{\partial \frac{f(p|c')}{f(p|c)}}{\partial p} dc'.$$

Restricting attention to prices in the interval  $[p(\bar{c}), \bar{p}]$ , we have that

$$\frac{\partial \frac{f(p|c')}{f(p|c)}}{\partial p} = \frac{\bar{p} - c'}{\bar{p} - c} \frac{N}{N-1} \frac{c - c'}{(p - c')^2} \left( \frac{p - c}{p - c'} \right)^{\frac{1}{N-1}},$$

such that we obtain

$$\begin{aligned} \frac{\partial \delta(c|p)}{\partial p} &= - \frac{[\delta(c|p)]^2}{g(c)} \int_{\underline{c}}^{\bar{c}} g(c') \frac{\bar{p} - c'}{\bar{p} - c} \frac{N}{N-1} \frac{c - c'}{(p - c')^2} \left( \frac{p - c}{p - c'} \right)^{\frac{1}{N-1}} dc' \\ &= - \frac{[\delta(c|p)]^2}{g(c)} \frac{N}{N-1} \frac{(p - c)^{\frac{1}{N-1}}}{\bar{p} - c} \int_{\underline{c}}^{\bar{c}} g(c') \frac{(c - c')(\bar{p} - c')}{(p - c')^{\frac{2N-1}{N-1}}} dc'. \end{aligned}$$

Since  $\delta(c|p)$  is a density function for every  $p$  we have that  $\int_{\underline{c}}^{\bar{c}} \delta(c|p) dc = 1$  for all  $p$  and consequently,

$$\frac{\partial [\int_{\underline{c}}^{\bar{c}} \delta(c|p) dc]}{\partial p} = \int_{\underline{c}}^{\bar{c}} \frac{\partial \delta(c|p)}{\partial p} dc = 0. \quad (25)$$

This in turn implies that  $\frac{\partial \delta(c|p)}{\partial p}$  can neither be positive nor negative for all values of  $c$ .

In particular, since  $\delta(c|p)$  is continuously differentiable, it follows that for all prices  $p \in [p(\bar{c}), \bar{p}]$  there exists (at least) one cost level  $\hat{c}$  such that

$$\frac{\partial \delta(\hat{c}|p)}{\partial p} = 0.$$

Consequently, at this cost level

$$\int_{\underline{c}}^{\hat{c}} g(c') \frac{(\hat{c} - c')(\bar{p} - c')}{(p - c')^{\frac{2N-1}{N-1}}} dc' = 0.$$

For notational simplicity, let us introduce the function

$$\phi(p, c) = \int_{\underline{c}}^{\bar{c}} g(c') \frac{(c - c')(\bar{p} - c')}{(p - c')^{\frac{2N-1}{N-1}}} dc',$$

such that the above statement boils down to

$$\phi(p, \hat{c}) = 0.$$

To prove the lemma, it basically remains to show that (i) there exists only one unique  $\hat{c}$  that satisfies  $\phi(p, \hat{c}) = 0$ ; and (ii)  $\phi(p, c) < 0$  for  $c < \hat{c}$  and  $\phi(p, c) > 0$  for  $c > \hat{c}$ . The last part is due to the fact that  $\partial \delta(c|p)/\partial p$  and  $\phi(p, c)$  have opposing signs.

Assume that there exist more than one values of  $\hat{c}$  that satisfy  $\phi(p, \hat{c}) = 0$ . In such a case, at least one of these cost levels would have to satisfy

$$\frac{\partial \phi(p, c)}{\partial c} \Big|_{c=\hat{c}} \leq 0.$$

This, however, cannot be true since

$$\frac{\partial \phi(p, c)}{\partial c} = \int_{\underline{c}}^{\bar{c}} g(c') \frac{(\bar{p} - c')}{(p - c')^{\frac{2N-1}{N-1}}} dc' > 0.$$

Consequently, there can only be a unique cost level  $\hat{c}$  that satisfies  $\phi(p, \hat{c}) = 0$ , and  $\phi(p, c) < 0$  for  $c < \hat{c}$  and  $\phi(p, c) > 0$  for  $c > \hat{c}$  obtain trivially. Thus,

$$\frac{\partial \delta(c|p)}{\partial p} \begin{cases} > 0 & \text{if } c < \hat{c} \\ = 0 & \text{if } c = \hat{c} \\ < 0 & \text{if } c > \hat{c}. \end{cases}$$

For  $p' > p$  the posterior  $\delta(c|p')$  puts more weight on low values of  $c$  and less weight on high values of  $c$  as compared to  $\delta(c|p)$ . Put differently,  $\delta(c|p)$  first order stochastically dominates  $\delta(c|p')$ . ■

**Proof of Lemma 3.3.** Define  $\rho_2(\bar{p})$  as the (hypothetical) price at which a consumer in round two would be indifferent between buying at that price and continuing to search after having observed  $\bar{p}$  in the first search round. There are two cases to consider: (i)  $\rho_2(\bar{p}) < \rho = \bar{p}$ ,

and (ii)  $\rho_2(\bar{p}) \geq \rho_1 = \rho = \bar{p}$ . In the remainder of this proof, we argue that case (i) leads to an inconsistency, while case (ii) leads to a consistent procedure with  $\rho$  being indeed defined by

$$v - \rho = v - E(p|\rho) - s.$$

CASE (i): Assume that  $\rho_2(\bar{p}) < \rho$ , and let us introduce for notational simplicity  $\hat{p} = \rho_2(\bar{p})$ . Note that if a consumer observes  $\hat{p}$  in the first search round, she would prefer to buy rather than continue to search. Formally, we have that  $v - \hat{p} \geq \pi^s(\hat{p}; N - 1)$ , where we denote by  $\pi^s(\hat{p}; N - 1)$  the payoff of a consumer who has observed  $\hat{p}$  in the first search round and continues to search optimally given that there are potentially still  $N - 1$  other firms to sample.

Next consider the hypothetical situation that there are in total  $N + 1$  firms in the market, but all of these set their prices according to the equilibrium price distribution in the market with  $N$  firms. Assume that a consumer has already observed two prices,  $\hat{p}$  and  $\bar{p}$ . If this consumer continues to search optimally after this hypothetical situation, her payoff is given by  $\pi^s(\hat{p}, \bar{p}; N - 1)$ . Furthermore, denote by  $\pi^s(\hat{p}, \bar{p}; N - 2)$  the payoff if the consumer continues to search optimally after having observed  $\hat{p}$  and  $\bar{p}$  and there are potentially only  $N - 2$  other firms to sample, as is true in our original market with  $N$  firms. Note further that, since  $\rho_2(\bar{p}) = \hat{p} < \rho$ , we have that  $v - \hat{p} = \pi^s(\hat{p}, \bar{p}; N - 2)$ .

At this stage, note that the benefits of search as defined above must satisfy

$$\pi^s(\hat{p}; N - 1) > \pi^s(\hat{p}, \bar{p}; N - 1) \geq \pi^s(\hat{p}, \bar{p}; N - 2).$$

The second inequality is obvious: a consumer can never get a higher payoff of searching if she has the same price observations in her pocket and she has fewer search alternatives left. Regarding the first inequality, note that  $\delta(c|\hat{p}, \bar{p})$  is the posterior distribution which is obtained by updating the belief  $\delta(c|\hat{p})$  using the price observation  $\bar{p}$ . As  $\bar{p} > \underline{p}(\bar{c})$  we can apply (a modified version of) Lemma 3.1, taking  $\delta(c|\hat{p})$  instead of  $g(c)$  as prior belief distribution, to obtain that  $\delta(c|\hat{p}, \bar{p})$  FOSD  $\delta(c|\hat{p})$ . Hence, consumers become more pessimistic about the underlying cost level having observed the price  $\bar{p}$ . Further, recall that we have that  $F(p|c)$  FOSD  $F(p|c')$  whenever  $c > c'$ , such that consumers expect higher prices when they expect higher costs. Consequently, it is strictly less attractive to continue searching after having observed both  $\hat{p}$  and  $\bar{p}$  compared to a situation where only  $\hat{p}$  had been observed. Thus, we arrive at an inconsistency, because

$$v - \hat{p} \geq \pi^s(\hat{p}; N - 1) > \pi^s(\hat{p}, \bar{p}; N - 2) = v - \hat{p}.$$

CASE (ii): now assume that  $\rho_2(\bar{p}) \geq \rho = \bar{p}$ . A consumer who has observed the upper bound  $\rho$  and continues to search will now buy in the next period at a price below  $\rho$  with probability one. Thus, a consumer who has observed the upper bound  $\bar{p}$  and continues to search optimally

will get a payoff of  $\pi^s(\bar{p}; N-1) = v - E(p|\bar{p}) - s$ . It follows that  $v - \rho = v - E(p|\rho) - s$ . ■

**Proof of Proposition 3.3.** For a PBERP to exist, a consumer must necessarily find it optimal to buy if she observes a price lower than the reservation price  $\rho$ , and firms must not make negative profits when choosing prices from the PBERP price distribution. In the following, we provide restrictions on the model's parameters such that these two conditions are satisfied. To this end, we proceed in two steps. In Part 1, we first show that at prices  $p$  with  $\underline{p}(\bar{c}) < p < \rho$  an uninformed consumer prefers to buy instead of continuing to search and buy necessarily in the next round. Later on, we allow for more general search behaviors. In Part 2, we provide conditions such that a consumer also finds it optimal to buy when she observes a price in  $[\underline{p}(\underline{c}), \underline{p}(\bar{c})]$ . Finally, we show that these conditions already guarantee that firms make positive profits in equilibrium.

PART 1: Consider for the time being the following hypothetical scenario. A consumer observes the price  $p'$  and must decide between buying at  $p'$  immediately and visiting one more firm, provided that she must *necessarily* buy after having obtained this additional price quote. In such a situation, the consumer is indifferent between buying and searching if she observes the reservation price, i.e. if  $p' = \rho$ , as this price equates her net benefits of search to zero. Now assume that the consumer has observed a lower price  $p'$  in  $(\underline{p}(\bar{c}), \rho)$ . By Lemma 3.2, she is now more pessimistic about the cost and thus the possibility of finding lower prices compared to if she had observed  $\rho$ . Consequently, she must find it optimal to buy rather than search one more firm.

So far, we have argued that for prices  $p$  such that  $\underline{p}(\bar{c}) < p < \rho$ , the uninformed consumer prefers to buy instead of continuing to search and buy necessarily in the next round. We now consider more general search behaviors. In particular, it may easily be the case that the consumer, after continuing to search, may not want to buy after observing the next price, but instead prefers to continue searching at least one more time. We will now show that this cannot be optimal either if consumers observe prices  $p$  with  $\underline{p}(\bar{c}) < p \leq \rho$ .

If a consumer has observed  $t$  prices with  $p' = \min(p_1, \dots, p_t) \geq \underline{p}(\bar{c})$ , then her payoff from searching is given by

$$v - s - F(p'|p_1, \dots, p_t) \cdot \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p_1, \dots, p_t) dpdc - (1 - F(p'|p_1, \dots, p_t))p'$$

where  $F(p'|p_1, \dots, p_t) = \int_{\underline{c}}^{\bar{c}} F(p'|c) \delta(c|p_1, \dots, p_t) dc$  is the subjective probability of finding a price lower than  $p'$  of a consumer who has observed prices  $p_1, \dots, p_t$ . By Bayes rule, we have

that:

$$\delta(c|p_1, \dots, p_t) = \frac{\delta(c|p_1, \dots, p_{t-1})f(p|c)}{\int_{\underline{c}}^{\bar{c}} \delta(c'|p_1, \dots, p_{t-1})f(p|c')dc'}. \quad (26)$$

In this sense,  $\delta(c|p_1, \dots, p_t)$  is the distribution obtained from updating  $\delta(c|p_1, \dots, p_{t-1})$  after the price observation  $p_t$ . We can again apply (a modified version of) Lemma 3.1, taking  $\delta(c|p_1, \dots, p_{t-1})$  instead of  $g(c)$  as prior distribution, to obtain that if  $p_t \geq \underline{p}(\bar{c})$ , then  $\delta(c|p_1, \dots, p_t)$  first order stochastically dominates  $\delta(c|p_1, \dots, p_{t-1})$ . By induction,  $\delta(c|p_1, \dots, p_t)$  FOSD  $\delta(c|p')$ . Thus, as  $E(p|c)$  is increasing in  $c$  and  $F(p'|c)$  is decreasing in  $c$ , we have that

$$\begin{aligned} v - s - F(p'|p_1, \dots, p_t) \cdot \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p_1, \dots, p_t) dc - (1 - F(p'|p_1, \dots, p_t))p' \\ < v - s - F(p'|p_1, \dots, p_t) \cdot \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p') dc - (1 - F(p'|p_1, \dots, p_t))p' \\ < v - s - F(p') \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p') dc - (1 - F(p'))p', \end{aligned}$$

where the last inequality follows from the fact that  $E(p|p < p', c) < p'$  and that  $F(p'|p_1, \dots, p_t) < F(p')$ . Thus, as

$$v - p' > v - s - \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p') dc - (1 - F(p'))p',$$

it follows that if  $p' = \min(p_1, \dots, p_t) \geq \underline{p}(\bar{c})$

$$v - p' > v - s - F(p'|p_1, \dots, p_t) \cdot \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p_1, \dots, p_t) dc - (1 - F(p'|p_1, \dots, p_t))p'.$$

Consequently, the consumer does not want to continue searching and then buy immediately in the next round after having observed  $t$  prices with  $\underline{p}(\bar{c}) \leq p' = \min(p_1, \dots, p_t) \leq \rho$  for any  $t$ .

Let us finally consider the following, alternative search strategy: the consumer decides to continue searching in round  $t$  and, after having observed one more price, does not buy at any of the prices observed up to that moment if the newly observed price is larger than  $\underline{p}(\bar{c})$ . It is easy to see that, if the consumer searches in this way and then buys at a later moment, her payoff evaluated from period  $t$  onwards is smaller than

$$v - s - F(\tilde{p}|p_1, \dots, p_{t+1}) \int_{\underline{c}}^{\bar{c}} E(p|p < \tilde{p}, c) \delta(c|p_1, \dots, p_{t+1}) dc - (1 - F(\tilde{p}|p_1, \dots, p_{t+1}))\tilde{p}$$

for some  $\tilde{p} \geq \underline{p}(\bar{c})$ . Using the argument given above, it follows that this is not optimal as well. By induction, it follows that it is also not optimal to wait more than one period. Taken together, the arguments we have used so far show that a consumer will indeed buy if she observes a price in the interval  $(\underline{p}(\bar{c}), \rho)$ .

PART 2: Let us now consider consumer behavior if a price below  $\underline{p}(\bar{c})$  is observed. By assumption, we have  $\underline{p}(\bar{c}) \leq \rho^k(\underline{c})$ , so that all these prices are below the reservation price in a model where (i) the consumers are informed about the cost realization and (ii) know this realization is equal to  $\underline{c}$ . We will argue that consumers should always buy at such prices.

As we consider prices  $p' \leq \rho^k(\underline{c})$ , it easily follows that

$$\begin{aligned} v - p' &\geq v - \rho^k(\underline{c}) = v - s - E^k(p|\underline{c}) \\ &> v - s - F^k(p'|\underline{c})E^k(p|p < p', \underline{c}) - (1 - F^k(p'|\underline{c}))p' \\ &\geq v - s - F(p') \int_{\underline{c}}^{\bar{c}} E(p|p < p', c) \delta(c|p') dc - (1 - F(p'))p'. \end{aligned}$$

Thus, the consumer prefers to buy at prices  $p' \leq \rho^k(\underline{c})$  instead of continuing to search and buy then immediately. Furthermore, arguments similar to the ones used in Part 1 of this proof can be applied to establish that it does neither pay off to continue searching and then not to buy immediately after observing some price. Thus, under our assumption  $\underline{p}(\bar{c}) \leq \rho^k(\underline{c})$ , a reservation price strategy is optimal for the consumer.

To complete the existence part of the proof, we need to rewrite the condition  $\underline{p}(\bar{c}) \leq \rho^k(\underline{c})$  in terms of the model's exogenous parameters and examine the profits made by firms given the consumers' behavior. Note first that we have

$$\begin{aligned} \underline{p}(\bar{c}) &= \frac{\lambda N}{\lambda N + 1 - \lambda} \bar{c} + \frac{1 - \lambda}{\lambda N + 1 - \lambda} \left( \frac{s}{1 - \alpha} + \int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc \right) \\ &< \bar{c} + \frac{1 - \lambda}{\lambda N + 1 - \lambda} \left( \frac{s}{1 - \alpha} \right), \\ \rho^k(\underline{c}) &= \underline{c} + \frac{s}{1 - \alpha} \end{aligned}$$

such that certainly  $\underline{p}(\bar{c}) \leq \rho^k(\underline{c})$  if

$$\bar{c} \leq \underline{c} + \frac{\lambda N}{\lambda N + 1 - \lambda} \left( \frac{s}{1 - \alpha} \right).$$

To check that firms' profits are positive, it is sufficient to check that for all  $c$ ,  $\underline{p}(c) > c$ . As  $\underline{p}(c) = \frac{\lambda N}{\lambda N + 1 - \lambda} c + \frac{1 - \lambda}{\lambda N + 1 - \lambda} \bar{p}$ , this is the case if  $\bar{p} > \bar{c}$ . As  $\bar{p} = \frac{s}{1 - \alpha} + \int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc \geq \frac{s}{1 - \alpha} + \underline{c}$ . This inequality is automatically satisfied if  $\bar{c} \leq \underline{c} + \frac{\lambda N}{\lambda N + 1 - \lambda} \left( \frac{s}{1 - \alpha} \right)$ . Uniqueness of the equilibrium is proved by showing that the reservation price is uniquely defined by

$$\bar{p} = \frac{s}{1 - \alpha} + \int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc.$$

To show that this equation has a unique solution, we show that the RHS is decreasing in  $\bar{p}$ , which together with the fact that the LHS is increasing in  $\bar{p}$ , suffices. For this purpose, we have

to evaluate the sign of

$$\frac{\partial \delta(c|\bar{p})}{\partial \bar{p}} = -\frac{[\delta(c|\bar{p})]^2}{g(c)} \int_{\underline{c}}^{\bar{c}} g(c') \frac{\partial \frac{f(\bar{p}|c')}{f(\bar{p}|c)}}{\partial \bar{p}} dc'.$$

We have that

$$\frac{f(\bar{p}|c')}{f(\bar{p}|c)} = \left( \frac{\bar{p} - c}{\bar{p} - c'} \right)^{\frac{1}{N-1}}$$

and therefore

$$\frac{\partial \frac{f(\bar{p}|c')}{f(\bar{p}|c)}}{\partial \bar{p}} = \frac{1}{N-1} \frac{c - c'}{(\bar{p} - c')^2} \left( \frac{\bar{p} - c}{\bar{p} - c'} \right)^{\frac{2-N}{N-1}}.$$

It follows that if  $\frac{\partial \frac{f(\bar{p}|c')}{f(\bar{p}|c)}}{\partial \bar{p}} > 0$  for some  $\tilde{c}$  it is positive for all  $c > \tilde{c}$ . Moreover, since  $\delta(c|\bar{p})$  is a density function we have  $\int_{\underline{c}}^{\bar{c}} \delta(c|\bar{p}) dc = 1$  and consequently we have

$$\frac{\partial [\int_{\underline{c}}^{\bar{c}} \delta(c|\bar{p}) dc]}{\partial \bar{p}} = \int_{\underline{c}}^{\bar{c}} \frac{\partial \delta(c|\bar{p})}{\partial \bar{p}} dc = 0. \quad (27)$$

This in turn implies that

$$\frac{\partial \delta(c|\bar{p})}{\partial \bar{p}} \begin{cases} > 0 & \text{if } c < \hat{c} \\ = 0 & \text{if } c = \hat{c} \\ < 0 & \text{if } c > \hat{c}. \end{cases}$$

Thus, the posterior  $\delta(c|\bar{p})$  puts relatively more weight on low values of  $c$  the larger the values  $\bar{p}$ . Thus, the RHS is decreasing in  $\bar{p}$ . ■

**Proof of Proposition 3.4.** In a reservation price equilibrium, there is some  $\rho'$  such that consumers buy in the first round of search if, and only if,  $p \leq \rho'$ . It is clear that  $\rho' \geq \rho^k(\underline{c})$  as otherwise consumers will buy even if they observe a price (slightly) above  $\rho'$ . Also,  $\rho' \leq \rho$ . This latter claim follows from the following observations. First, for all  $\bar{p} \leq \rho^k(\underline{c})$ ,  $v - \bar{p} < v - s - E(p|\bar{p})$ , i.e., if the upper bound of the price distribution is relatively low consumers would prefer to buy at the upper bound rather than continuing to search. Second, for all  $\bar{p} \geq \rho^k(\underline{c})$ ,  $v - \bar{p} > v - s - E(p|\bar{p})$ , i.e., if the upper bound of the price distribution is relatively high consumers would prefer to continuing to search if they observe a price equal to the upper bound rather than buy. Third,  $\rho$  is uniquely defined by  $v - \rho = v - s - E(p|\rho)$ . Thus, for all  $\rho' > \rho$ ,  $v - \rho > v - s - E(p|\rho)$ , i.e., consumer prefer to buy immediately instead of continuing to search. Thus, it follows that the upper bound of the price distribution  $\bar{p} = \rho'$  and  $\rho^k(\underline{c}) \leq \bar{p} \leq \rho$ .

Let us then consider the profits of firms. In an equilibrium it has to be the case that these profits are positive for all  $c$ . This is the case if, and only if, for all  $c$ ,  $\underline{p}(c) > c$ . As

$\underline{p}(c) = \frac{\lambda N}{\lambda N + 1 - \lambda} c + \frac{1 - \lambda}{\lambda N + 1 - \lambda} \bar{p}$ , this is the case if, and only if,  $\bar{p} > \bar{c}$ . A reservation price equilibrium therefore does not exist if  $\bar{p} = \frac{s}{1 - \alpha} + \int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc < \bar{c}$ . This is the case if (i)  $s$  is relatively small enough or (ii)  $\bar{c}$  and  $\underline{c}$  are relatively far apart and  $g(c)$  has a relatively high probability mass close to  $\underline{c}$ . ■

**Proof of Proposition 4.1.** By equations (9), (10), (20), and (21) it is obvious that both  $E(p) > E^k(p)$  and  $E(p_\ell) > E^k(p_\ell)$  hold if  $E(c|\bar{p}) = \int \delta(c|\bar{p}) dc > E(c) = \int g(c) dc$ . By Lemma 3.1,  $\delta(c|\bar{p})$  FOSD  $g(c)$ , such that  $E(c|\bar{p}) > E(c)$  obtains trivially. As the expected price of each consumer type is higher under incomplete information, expected profits are higher and thus producer welfare is higher. Furthermore, as consumer welfare is inversely related to the expected price, we have that consumer welfare of both shoppers and non-shoppers is lower under incomplete information. ■

**Proof of Proposition 4.2.** Under incomplete information, each firm's conditionally expected profit  $E(\Pi|c)$  amounts to

$$E(\Pi|c) = \frac{1 - \lambda}{N} (\bar{p} - c) = \frac{1 - \lambda}{N} \left( \frac{s}{1 - \alpha} + \int_{\underline{c}}^{\bar{c}} c \delta(c|\bar{p}) dc - c \right).$$

To see this, note that in a mixed strategy equilibrium each firm's expected profit must be equal to the profit resulting from charging the upper bound of the price distribution. Obviously,  $E(\Pi|c)$  is a decreasing function of the cost realization  $c$ . Under complete information, we have that

$$E^k(\Pi|c) = \frac{1 - \lambda}{N} \frac{s}{1 - \alpha}$$

which clearly is independent of  $c$ . Finally, note that the difference in profits between is given by

$$E(\Pi|c) - E^k(\Pi|c) = \frac{1 - \lambda}{N} \left( c - \int_{\underline{c}}^{\bar{c}} c' \delta(c'|\bar{p}) dc' \right),$$

establishing that conditional expected profits under incomplete information are higher for low cost realizations, and lower for high cost realizations, compared to conditional expected profits under complete information. ■

**Proof of Proposition 4.3** Janssen and Moraga-González (2004) have shown that  $\alpha$  is increasing in  $N$  and that  $\alpha \rightarrow 1$  as  $N \rightarrow \infty$ . Consider the reservation price as defined in (16)

$$\rho = E(c|\bar{p}) + \frac{s}{1 - \alpha}.$$

Note that we have that  $E(c|\bar{p}) \in [\underline{c}, \bar{c}]$ . Hence, we have that  $\rho \rightarrow \infty$  as  $N \rightarrow \infty$ . Now, consider  $E(c|\bar{p}) = \int_{\underline{c}}^{\bar{c}} \delta(c|\bar{p})cdc$ . We have that

$$\delta(c|\bar{p}) = \frac{g(c)f(\bar{p}|c)}{\int_{\underline{c}}^{\bar{c}} g(c')f(\bar{p}|c')dc'} = \frac{g(c)}{\int_{\underline{c}}^{\bar{c}} g(c')\frac{f(\bar{p}|c')}{f(\bar{p}|c)}dc'}.$$

Further, note that

$$\frac{f(\bar{p}|c')}{f(\bar{p}|c)} = \left(\frac{\bar{p}-c}{\bar{p}-c'}\right)^{\frac{1}{N-1}}.$$

As we have that  $\bar{p} = \rho \rightarrow \infty$  as  $N \rightarrow \infty$  it follows that the previous expression converges to one. This in turn implies that the posterior distribution  $\delta(c|p)$  converges to the prior distribution  $g(c)$  and we have that

$$\lim_{N \rightarrow \infty} (E(c|\bar{p}) - E(c)) = 0.$$

From equations (5), (16), (9) (20), (10), and (21) it follows that,

$$\begin{aligned} \rho - E(\rho^k(c)) &= E(c|\bar{p}) - E(c), \\ E(p) - E^k(p) &= \alpha[E(c|\bar{p}) - E(c)], \quad \text{and} \\ E(p_\ell) - E^k(p_\ell) &= (1 - \alpha)[E(c|\bar{p}) - E(c)]. \end{aligned}$$

It follows that the above expressions converge to 0 as  $N \rightarrow \infty$ . ■