Estimating the State Vector of Linearized DSGE Models without the Kalman Filter

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Abstract: This note presents a simple method for estimating the state vector of linearized DSGE models without using the Kalman filter. The conditional covariance matrix of the state vector is also derived. The method can easily cope with filtered data, and with arbitrary patterns of missing observations.

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This note presents a simple method for estimating the state vector of linearized dynamic stochastic general equilibrium (DSGE) models, without using the Kalman filter. The method can easily handle filtered data and arbitrary patterns of missing observations.

The solution of generic linearized DSGE models can be represented as

\[ x_t = h_t \cdot x_{t-1} + \eta_t \cdot \varepsilon_t, \]

where \( x_t \) is an \( n_x \times 1 \) vector of (possibly unobserved) endogenous and exogenous state variables, while \( \varepsilon_t \) is a white noise vector that is orthogonal to \( x_s \) for \( s < t \). \( h_t \) and \( \eta_t \) are matrices whose coefficients are functions of structural model parameters; see, e.g., Schmitt-Grohé and Uribe (2004, 2011). Assume that \( x_t \) is stationary, and that \( \varepsilon_t \) is normally distributed.

Let \( X = [x_1; x_2; \ldots; x_T] \) denote the column vector obtained by stacking the state vectors for periods \( t=1,\ldots,T \). Date \( t \) observables are given by the \( n_y \times 1 \) vector \( y_t = g_t \cdot X \),
with \( n_y < n_x \), \( g_t \) is a \( n_y \times (n_x \cdot T) \) vector of deterministic (possibly time-varying) coefficients; thus \( y_t \) is a sub-set (or linear combination) of the state vector \( X \). This specification encompasses cases in which observables \( y_t \) just depend on the date \( t \) state vector, \( y_t = g_t \cdot x_t \), but also captures cases in which the data is filtered using one-sided or two-sided filters (such as the HP filter) so that \( y_t \) also depends on \( x_s \) for \( s \neq t \). Let \( Y = [y_1; y_2; \ldots; y_T] \) denote the stacked vector of observables for \( t=1,\ldots,T \).

This note shows how to estimate the (stacked) state vector \( X \), given data \( Y \), very simply, without using the Kalman filter. Estimates of (latent) states are often of great interest for economic analysis.

The method is inspired by Schmitt-Grohé and Uribe (2011) who show how to construct the likelihood of \( Y \) without using the Kalman filter, by interpreting \( Y \) as a draw from a multivariate density.\(^1\)

Observe that the ‘smoothed’ estimate of \( X \), i.e. the conditional expectation of \( X \) given \( Y \), is

\[
E[X|Y] = E(XY') [E(YY')^{-1}] \cdot Y,
\]

due to the joint normality of \( X, Y \).

Note that \( Y = G \cdot X \), where \( G = [g_1; g_2; \ldots; g_T] \). Thus, \( E(YY') \) and \( E(XY') \) can be computed as follows: \( E(YY') = GE(XX')G' \), \( E(XY') = E(XX')G' \), with

\[
E(XX') = \begin{bmatrix}
E_{x_1 x_1} & \ldots & E_{x_1 x_T} \\
\ldots & \ldots & \ldots \\
E_{x_T x_1} & \ldots & E_{x_T x_T}
\end{bmatrix},
\]

where \( E_{x_s x_t} = (h)^{s-t} \Sigma_x \) for \( s > t \) and \( E_{x_s x_t} = (h)^{t-s} \Sigma_x \) for \( s \leq t \), with \( \Sigma_x = E(x_s x_t') \).

The smoothed estimate of the vector of innovations \( \epsilon_t \) can be computed using

\[
E[\epsilon_t|Y] = (\eta' \cdot \eta)^{-1} \cdot \eta' \cdot (E[x_t|Y] - h \cdot E[\epsilon_{t-1}|Y]),
\]

\(^1\)As pointed out by Schmitt-Grohé and Uribe (2011), the (recursive) Kalman filter cannot be employed when the data are filtered using a two-sided filter.
The conditional covariance matrix of \( X \) (useful for computing confidence sets for the state variables) is
\[
E[XX\mid Y] = E(XX') - E(XY')\{E(YY')\}^{-1}E(YX'),
\]

Estimates of the state vector \( X \) based on smaller data sets (e.g. due to missing observations) can be computed by adapting the above formulae. Let \( Y^A = W\cdot Y \) be a vector consisting of a subset of the elements of \( Y \), where \( W \) is a \( n_w \times (n_rT) \) ‘selection’ matrix of zeros and ones with \( n_w < n_rT \). Then
\[
E[X \mid Y^A] = E(XY')W'[WE(YY')W']^{-1}Y^A
\]
and
\[
E[XX' \mid Y^A] = E(XX') - E(XY')W'[WE(YY')W']^{-1}WE(YX').
\]

Computer code that implements the algorithm is available on the author’s web page.

References

------, 2011. Evaluating the sample likelihood of linearized DSGE models without the use of the Kalman filter, Economics Letters 109, 142-143.