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ECORE DISCUSSION PAPER

2010/12

Stock Prices, Anticipations and Investment in General Equilibrium

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CORE DISCUSSION PAPER
2009/83

**Stock prices, anticipations and investment
in general equilibrium**

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December 2009

Abstract

We propose an objective for the firm in a model of production economies extending over time under uncertainty and with incomplete markets. We derive the objective of the firm from the assumption of initial-shareholders efficiency. Each shareholder is assumed to communicate to the firm her marginal valuation of profits at all future events (expressed in terms of initial resources). In defining her own marginal valuation of the firm's profits, a shareholder takes into consideration the direct impact of a change in the value of dividends but also the impact of future dividends on the firm's stock price when she trades shares. To predict the impact on the stock price, she uses a state price process, her price theory. The firm computes its own shadow prices for profits at all date-events by simply adding up the marginal valuations of all its initial shareholders. If no restrictions are placed on individual price theories, the existence of equilibria may require financial constraints on a firm's investment when its shareholders are more optimistic than the market about the profitability of such investment. We then impose that price theories be compatible with the observed equilibrium: they should satisfy a no-arbitrage condition. We show by means of an example that, with incomplete markets and no-short selling constraints, this restriction on price theories is not enough to bring consistency in the individuals' marginal evaluations: a financial constraint on the firm's investment may still be needed to obtain an equilibrium.

Keywords: general equilibrium, incomplete markets, stock prices, anticipations, investment constraints, arbitrage.

JEL Classification: D2, D51, D52, D53, G11, G12, G32, M20, P12

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We would like to thank Jean-Marc Bonnisseau for numerous, detailed, important comments, and seminar audiences at Kos, CORE, Paris 1, Torino, Salerno, HIS

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the authors.

Foreword

Investment decisions are often made today by agents who will not themselves experience all the implications of their decisions. This is a central issue for long term investments like pollution abatement. It is also an issue for business investments, because ownership of firms evolves over time. This feature raises a conceptual issue under incomplete markets. If the current shareholders of a firm plan trading shares in the future, how will they evaluate alternative profit streams?

Under complete markets, production plans (including investments) have a well-defined present value, and maximisation of that present value serves the interests of all shareholders, present and future. Under incomplete markets, *alternative plans* are not valued uniquely by the market, and shareholders will be concerned with the implications of such plans for the prices at which they will *trade shares* in the future. That is, they will be concerned with the *derivatives* of stock prices with respect to future profits: how will the stock price of the firm at time t respond to ulterior profits?

One answer to this question has been proposed in the seminal paper by Grossman and Hart (1979); namely, that shareholders should equate these derivatives with their own marginal rates of substitution for income. We bring out an element of inadequacy in this answer: under incomplete markets, *personal* marginal rates of substitution need not correspond to *market responses*. In general, current shareholders must *evaluate* the market responses, using all the information provided by observables plus (possibly) their private information and/or subjective beliefs. We label these evaluations *price theories* and we retain the Grossman and Hart specification of *competitive price perceptions*, which leads to define price theories as vectors of coefficients.

This paper introduces an equilibrium concept that embodies two features: (i) every firm's current decision is Pareto efficient for the firm's current share-

¹We would like to thank Jean-Marc Bonnisseau for numerous, detailed, important comments, and seminar audiences at Kos, CORE, Paris I, Torino, Salerno, IHS Vienna, Essex and Roma for useful discussions and suggestions.

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holders; (ii) current shareholders may hold idiosyncratic and firm - specific price theories. Assuming that these price theories are upper hemi-continuous in observables, we show that existence of an equilibrium may require bounds on the ability of a firm to raise investment capital, when there is excess supply of shares at a zero price (when the proposed investment is underfinanced). This is a natural feature: it is repeatedly observed on financial markets, reflecting the possibility that a firm's owners are more optimistic than the market about the profitability of their investment projects. There is no alternative *general* way for the market to sanction excessive optimism (or, for that matter, to cope with excess supply at a zero price). We thus obtain a general existence result for *investment-constrained equilibria*, general in the sense of minimal assumptions on price theories (Theorem 2).

We investigate next whether observables might constrain price theories to the extent of forestalling the underfunding of investments. Such is the case under *strict no-arbitrage*, a standard concept in finance. But strict no-arbitrage is not guaranteed when short sales are ruled out, as assumed here. In that more realistic case, the weaker concept of *no unlimited arbitrage* is appropriate. We show by means of one example (Example 1 in section 4) that, even if the price theories of all shareholders are required to satisfy no unlimited arbitrage, constraints on investments may still be needed to achieve an equilibrium. We then prove that a sufficient condition to obviate the need for financial constraints in a two-period model is that all current shareholders of a given firm *agree* about a price theory compatible with no unlimited arbitrage (Theorem 3). But that special result does not extend to longer horizons; the scope for obviating the need of financial constraints through natural restrictions on price theories is limited.

This paper consists of: (i) an extensive non-technical introduction, reviewing the problem and the previous literature (sections 1.1 and 1.2) and summarising our contribution (1.3); (ii) a description of our model (section 2) and a definition of equilibrium satisfying initial shareholder efficiency (section 3); (iii) a definition and existence theorem for investment-constrained equilibrium, and some implications of limited arbitrage (section 4); (iv) a brief section of concluding comments (section 5); (vi) an Appendix collecting proofs. Sections 1 and 2 - 4 are self-contained and can be read independently.

1 Subject-matter and overview

1.1 General equilibrium and incomplete markets

The purpose of this paper is to contribute to the theory of general equilibrium in production economies extending over time under uncertainty and with incomplete markets.

The standard model fitting these specifications is known as GEI: General Equilibrium with Incomplete markets; see, e.g. Geanakoplos (1990) or Magill and Shafer (1991) for surveys. The basic specification rests on that in Chapter 7 of Debreu (1959). The economy consists of two kinds of agents: consumers and firms. Time and uncertainty are captured by an event tree that specifies, for each date up to a finite horizon, the set of possible date-events reflecting the (common) information of the agents at that date. There are L physical commodities at each date-event. With N date-events over the tree, the commodity space is the NL -dimensional Euclidean space. Each consumer (household) h is defined by her consumption set in that space, by her initial endowment of commodities in the same space, and by her preferences among NL -dimensional consumption vectors. Each firm j is defined by its production set in the same space. In addition, all firms are initially owned by the consumers.

In Debreu's model, there exist markets at date 0 for trading all commodities (that is, for trading claims to each physical commodity contingent on each date-event). The resulting model is formally equivalent to that of a production economy extending over a single period: consumers face a single budget constraint, over which they maximise preferences; firms maximise the present value of profits at market prices. Under this complete market system, trading in shares of ownership is redundant: at equilibrium each firm is analogously defined by a vector of event-dependent profits, with present value equal to the firm's market value. Trading in contingent commodities is a perfect substitute for trading in shares of the firms. Uncertainty makes no difference, due to the perfect insurance opportunities provided by the complete markets. The set of competitive equilibria is also the same as in Debreu.

An alternative interpretation of the same model had appeared earlier in a seminal paper by Arrow (1953). Restrict trade in commodities to spot markets at each date event, and add markets at each date-event for elementary securities, each paying off in a specific successor date-event. Under perfect

foresight of future prices, the set of attainable allocations is the same as in Debreu.

The assumption of complete markets has long been recognised as unrealistic. In the real world, not all contingencies are amenable to perfect insurance. In the words of Magill and Quinzii (1996, p.4): " the ideal structure of markets in which everything is traded out in advance would involve prohibitively large transactions costs". What we encounter in practice is a sequence of spot markets on which commodities are exchanged (as with Arrow), together with a limited set of asset markets through which *limited* reallocation of resources over time and across date-events is possible (at variance with Arrow). The resulting model is labelled GEI. Compared with the complete-markets model, two new features emerge: (i) consumers are faced with multiple budget constraints; and (ii) firms are not able to evaluate production plans in terms of market values.

1.2 Investment under private ownership

1.2.1

The modeling of a production economy under uncertainty and with incomplete markets thus raises the issue of defining the decision criterion of the firms. This is the case even in the simplest, two-period setting. Production plans have a time 0 component, and $N - 1$ time 1 components (one per terminal date-event). There is a stock market open a time 0 , where shares of the firms are traded. A shareholder at time 0 after stock-market clearing receives a dividend equal to the value of the time 0 component of the production plan; in case of net investment by the firm, that dividend is negative (it operates like an addition to the stock price). At time 1 , the value of the production plan at a date-event accrues to the time 0 shareholders as a dividend. The stock market does not reopen (it would be redundant)¹. Consumers know the production plans of all firms when choosing their portfolios; and the stock market at time 0 clears through prices.

The new difficulty is that, under genuinely incomplete markets, the profits at time 1 under date-event s ($s = 1, \dots, N - 1$) need not have a well-defined market value at time 0 , when production plans are chosen. Indeed, with J firms, $J < N - 1$, only the J vectors of date 1 profits are priced by the market.

¹Indeed, the value of a firm would be equal to its dividends, and there would be no incentives to trade shares

Maximising the present value of profits (through choice of the production plan) is not well defined. Thus, one needs to specify a decision criterion for the firm. And the assumption of common, correct point expectations applies only to commodity prices at time 1 date-events. It does not apply to present values (at date 0) of event-specific profits, since there are no market prices for these.

Diamond (1967) introduced a notion of Constrained Pareto Optimality (CPO) for this model. An allocation is constrained feasible if and only if it is physically feasible, and *susceptible of being attained through asset trading at given production plans*. For normative purposes, it is easy to formulate necessary first-order conditions (FOC) on production plans required by CPO. To that end, remember that each consumer optimises its consumption subject to N distinct budget constraints. Denote by λ the N -vector of Lagrange multipliers associated with these constraints, and by $\bar{\lambda}$ the N -vector of ratios $\frac{\lambda_s}{\lambda_0}$. These define marginal rates of substitution between income at date 1 in date-event s and income at date 0. (Under the assumption of perfect foresight, these marginal rates of substitution are defined at common market-clearing spot prices for commodities at all date-events.) Consider a firm owned by a single consumer h , deciding simultaneously about her consumption and about the production plan of her firm. Then, jointly optimal consumption and production plans would entail that firm profits are maximal (over the production set) at the shadow prices $\bar{\lambda}^h$. For a general case in which the firm may have multiple shareholders, it is shown in Drèze (1974) that necessary FOC for CPO impose that *profits of each firm should be maximal at shadow prices defined as weighted averages of the marginal rates of substitution of the firm's shareholders, with weights given by respective shareholdings*. That is also a necessary FOC for Pareto efficiency of the production plan from the viewpoint of that firm's final shareholders. Pending that condition, there would exist changes in production and zero-sum transfers among shareholders making all of them better off. That result is an important clarification of the normative issues raised by market incompleteness in a production economy: it defines unambiguously a desirable² decision criterion for the firms, in a general model under standard assumptions. (It is often referred-to in the literature as the Drèze criterion.)

One interesting implication of the FOC for CPO is that, in general, there may not exist state-prices sustaining simultaneously the constrained-optimal

²Desirable because implied by necessary FOC for CPO.

plans of all firms. (For illustration, let two firms each be owned by a single shareholder, and let the shadow prices $\bar{\lambda}$ of these two consumers be different.) Of course, with limited trading in contingent claims, there is no reason why such state-prices should exist.

Drèze (1974) also brings out the important feature that, under incomplete markets, *the set of feasible allocations is not convex*. Indeed, the dividends received by a shareholder, which enter in her budget constraints, are defined as a product of two endogenous variables, namely the shareholding and the firm profits. This bilinearity results in a non-convex feasible set for the economy, the very set over which CPO is defined. Thus, *necessary* FOC are in general *not sufficient*. If equilibria are defined by the Drèze criterion, equilibria exist, but they need not be CPO³. It is shown in Geanakoplos et al. (1990) that, *generically* in initial endowments, they are *not* CPO⁴.

The relevance of this analysis for positive economics is limited. The notion of shareholder (Pareto) efficiency of production decisions is clearly appealing for privately owned firms or small partnerships. For large corporations listed on stock exchanges, the appeal is much less compelling: shareholder preferences (their $\bar{\lambda}$'s) have no natural channel of expression; the role of shareholders is limited to approval voting at general assemblies. It would take a lot of faith in the Coase theorem to claim realism for the Drèze criterion. However, an important step in the direction of realism is provided by later work of Drèze (1985, 1989 chapters 2 and 3)) on "equilibria of production and exchange", then "equilibria of production, exchange and labour contracts". The new ingredient is the so-called *control principle*: for each firm j , given shareholdings θ^{hj} , there exists a uniquely (endogenously) defined subset of controlling shareholders, say the Board of Directors; decisions about production plans must be endorsed by a majority of shareholders *including all the directors*. Thus, directors are veto players, a feature that also circumvents the Condorcet paradox of voting. And it is reasonable to assume that production decisions will be Pareto efficient for the small set of directors (subject to majority approval by all shareholders). Since the set of directors is endogenous (related to shareholdings), the general specification has undeniable realism. The only special assumption is that the correspon-

³Robust examples appear in Drèze (1974, section 4) - also for the nested case of complete markets.

⁴That result carries the momentous implication of potential Pareto improvements through departures from competitive market clearing; see Drèze and Gollier (1993) for an illustration.

dence defining the set of directors (as a function of shareholdings) is upper hemi-continuous for the discrete topology (see appendix 2 of Drèze 1989)⁵.

1.2.2

The characterisation of CPO as developed by Drèze is specific to a model extending over two periods. The extension to $T > 2$ periods appears in Bonisseau and Lachiri (2006). It is still the case that CPO requires firms to maximise profits at shadow prices obtained as weighted averages of the corresponding shadow prices (marginal rates of substitution) of their shareholders. But a new dimension arises: the shadow price for profits at date-event s should reflect the shadow prices of *shareholders at that date-event* - namely the shareholders to who these profits will accrue (either as dividends or as retained earnings incorporated in stock prices). Accordingly, implementing a decision criterion compatible with CPO calls for information not readily available; namely, the identity and shadow prices of future shareholders - some (many?) of which have no part in today's decisions! As we all know, there are many important economic decisions where concern for the interests of future stakeholders (future generations!) plays a determinant role: just think about climate change, exhaustion of natural resources or the public debt. In such contexts, the current generation must assess the implications of its current decisions for future stakeholders. That issue is receiving due attention. It is less widely appreciated that the very same feature is present in the daily operations of business firms, when markets are incomplete. That feature is also at the core of the present paper. In contrast with the social choice examples mentioned above, it is expected that current shareholders decide (choose investment and production plans) in light of their own interests. Ideally, one would hope that markets (here financial markets) bring these own interests in line with those of future shareholders: the evolution of the market value of a firm will reflect the interests of its future shareholders. It is in the interest of current shareholders to maximise the future value of the firm. But how to achieve this remains problematic.

Generically, the identity and holdings of shareholders will not be the same at a future date-event s as today. A household's shareholdings at s

⁵Examples of acceptable specifications include: all shareholders holding at least $\alpha\%$ of the shares, the n biggest shareholders, the leading shareholders owning at least $\beta\%$ of the shares. Also, there could be a single director (CEO), so the model encompasses a managerial theory of the firm

will depend upon the portfolio policy (share trading) of that household, as guided by the household's preferences and endowments on the one hand, by the production plans and market prices of the firms on the other hand. These portfolio policies are private information, devoid of public disclosure. And prospective shareholdings carry no well-defined voting rights today.

Actually, the problem already arises in the two-period model, if the production decisions are made by *initial* shareholders, prior to clearing of the stock market. These decisions are in the hands of one set of shareholders, but have implications for a (generically) different set. The $T > 2$ problem outlined in the previous paragraph is there! It is also the very problem motivating the present paper.

That problem was recognised for the first time in the important paper by Grossman and Hart ((1979) (GH)), who considered an arbitrary finite horizon. For simplicity, we discuss it here in the two-period framework, with production decisions finalised before clearing of the stock market. The gist of the new difficulty is this. When initial shareholders evaluate a firm's production plans, they take into account their own plans to trade shares; hence, they become concerned with *stock prices, not just dividends*. For a household planning to buy (sell) shares, a low (high) market price is a bonus. The attractiveness of firm j 's production plan for h is no longer defined by dividends alone; it also depends upon the impact of that production plan on the firm's market value. More specifically, the attractiveness for h of firm j 's profits in state s now depends on (i) dividends (multiplied by h 's final holdings in state s) and (ii) the impact of firm j 's profits in state s on the firm's market price today (multiplied by the number of shares that h plans trading). The second element is not directly observable, under incomplete markets. Household h is thus facing a new problem in forecasting: not only future prices matter, but also derivatives of today's market prices with respect to future profits. These derivatives could, in principle, be evaluated by computing (!) the impact of a change in the production plan of firm j on the equilibrium price vector, hence on the price of j . But this would amount to introducing an element of market power in the reasoning. Instead, GH introduced a new assumption, labeled "*competitive price perceptions*" (CPP), concerning the perceptions by consumers of the impact of additional profits at some future event s on the market values of firm j at earlier date-events. Let us represent these perceptions by $(S + 1)$ -vector, to be denoted α^{hj} for consumer h and called h 's *price theory*. What does it mean to entertain competitive price perceptions? Magill and Quinzii (1996, p.382) mention

two properties:

- "the price of a bundle of goods is the sum of the prices of its components;
- the unit price of each component is independent of the number of units of the good purchased or sold."

Applied to the forecasting of price changes on the stock market, where shares are bundles of goods defined by future profits, these principles imply that consumer h 's perceptions of price changes for the shares of firm j can be represented by a vector $\alpha^{hj} \in \mathbb{R}_{++}^{(S+1)}$ satisfying⁶:

$$\alpha_0^{hj} dq^j = \sum_{s \geq 0} \alpha_s^{hj} (p_s \cdot dy_s^j) \quad (1)$$

where dq^j is the anticipated variation of the market price of firm j at 0 and $p_s dy_s^j$ the change of its future profits at s .

The assumption of CPP in GH satisfies this definition⁷, but goes far beyond: these authors assume that h 's price theory, the same for all firms of which she is a shareholder, is given by h 's own marginal rates of substitution $\bar{\lambda}^h$. A trivial application reveals the shortcomings of that restrictive specification. If h is a consumer expecting to die (from a terminal illness) in state s leaving no heirs behind, λ_s^h could be zero. It would be preposterous for h to assume on that ground that additional profits at s will not be valued by the market at $t = 0$! Accordingly, we do not follow GH on that path (the justification by MQ on p.386 notwithstanding). It is unfortunate that two distinct assumptions, namely "competitive price perceptions", which is fine, and "price perceptions reflecting own consumption preferences" ("egocentric price perceptions"?) have been lumped under a single heading- initially by GH, but subsequently by the whole literature. As explained below, "egocentric price perceptions" have the momentous implication of cancelling from the evaluations of production plans the terms involving portfolio transactions.

⁶Price theories are allowed (not assumed!) to be firm specific because, as explained in 1.2.1, at a constrained-optimal solution there need not exist state-prices sustaining simultaneously the production plans of all firms.

⁷In this informal discussion we abstract from the presence of constraints on short selling.

1.3 Price theories and equilibrium

1.3.1

We are now (at long last!) ready to introduce the contribution of this paper to the theory of equilibria in production economies under uncertainty and incomplete markets. We propose an equilibrium concept based on initial-shareholders efficiency. We retain the assumption of perfect foresight, that is common, correct (at equilibrium), single-valued price expectations. We endow each household h with a set of price theories α^{hj} measuring the anticipated impact of future profits on today's stock prices while satisfying competitive price perceptions as discussed above, and we derive the decision criteria of firms by the principle of *initial shareholders efficiency*, to be defined presently.

The idea of initial shareholders efficiency is straightforward: the production plan chosen by firm j at date zero is such that there do not exist an alternative production plan and transfers of initial resources among the shareholders making all of them better off. This principle leads at once to the property that the value of the chosen production plan is maximal at shadow prices reflecting the marginal valuations of the shareholders (see the Proposition in section 2.2).

Each shareholder is assumed to communicate to firm j her marginal valuation of profits at future date events (a valuation expressed in terms of initial resources). Profits at s are distributed to final shareholders at s . In defining its own marginal valuation of firm j 's profits, a shareholder h will take two elements into consideration:

- profits (dividends) at date-event s are first valued at h 's shadow price ($\bar{\lambda}_s^h$) for resources at s , multiplied by the final (post-trade) shareholding of j by h , θ^{jh} ;
- next, h takes into account the impact of profits at s on the market value of j at node $t = 0$; this impact is assessed according to h 's price theory, and applied to the volume of h 's trade - thus with a positive sign in case of a sale, and a negative sign in case of a purchase.

The two elements are then added, and the sum defines h 's marginal valuation of firm j 's profits at date-event s , β_s^{hj} .

Note that these calculations rely on the consumption and portfolio plans of h . By themselves the shadow prices for resources $\bar{\lambda}^h$ do not convey the relevant information, if not combined with the planned portfolio trading plans

and the consumer price theories.

Now, firm j computes its own shadow prices for profits at all date-events, β^j , by simply adding up the marginal valuations of all its initial shareholders. There is no weighting involved, because shareholdings have been taken into account by the shareholders themselves in computing their own marginal valuations.

An equilibrium is a feasible allocation (x, θ, y) and prices (p for commodities, q for shares) such that all markets clear and all agents optimise: (x^h, θ^h) is best for h 's preferences subject to h 's budget constraints; y^j has maximal present value at the shadow prices β^j . Note that we do not enter the agents' price theories as elements of the equilibrium; we regard their formation (how they emerge from observations and expectations) as part of the primitives; see 1.3.3 infra.

We regard the proposed equilibrium concept as a first step, calling for the same extension that Drèze (1989) adds to Drèze (1974), namely an endogenous board of directors, But this first step is the natural stepping stone towards the more realistic extension.

1.3.2

The first analysis of existence of equilibrium in a temporal production economy with incomplete markets goes back to the seminal paper by Radner (1972). In this paper, the issue of the decision criterion of firms is eschewed by assuming existence of a primitive utility function for each firm (for its omnipotent manager). Radner then proves existence of a *pseudo-equilibrium*. If all stock prices are strictly positive, every pseudo-equilibrium is a full-fledged equilibrium. But situations may arise where the price of a firm is zero, and there is excess supply of shares. Such situations correspond to investment programs deemed profitable by the decision-maker(s) in the firm, but not by the market. They correspond to real-world situations where a firm offers new shares for subscription, but these are not fully subscribed by investors. In this paper, a reasonable decision criterion is introduced, namely efficiency for initial shareholders. Existence of a pseudo-equilibrium then follows from fairly standard assumptions. But the possibility of excess supply of shares at zero price remains open. This is a realistic possibility, resulting in a financial constraint on the investment policy of the firm. Accordingly, we define an *investment-constrained equilibrium* under which a firm's fund raising (negative dividend) is subject to a financing constraint, when there is excess supply

of shares at a zero price⁸. And we demonstrate the existence of such an equilibrium (Theorem 2) with arbitrary price theories – which also shows that no restrictions on price theories are implied at an investment - constrained equilibrium.

1.3.3

Theorem 2 places no restriction on individual price theories, beyond the natural requirement of upper hemi-continuity in observables. Under complete markets, price theories are given by market prices. Under incomplete markets, price theories should be compatible with observations. When the market prices of assets satisfy no-arbitrage, that property should be inherited by price theories. But there is no ground to impose strict no-arbitrage when short sales are excluded, as we assume here for the sake of realism. A weaker property, *no unlimited arbitrage*, is then relevant. We prove (Theorem 3) the existence of equilibria at which investment constraints are not binding when: (i) price theories satisfy no unlimited arbitrage, and (ii) all initial shareholders of a firm hold identical price theories. Unfortunately, that result does not extend to more periods. As noted above, the scope for obviating the need of financial constraints through natural restrictions on price theories is limited.

2 Model

The simplest model of an economy under uncertainty is one in which economic activity extends over two periods of time: $t = 0$, today, and $t = 1$, tomorrow. At $t = 1$, one of S mutually exclusive states of the world realizes. The state of the world is unknown at $t = 0$, when production and consumption plans are made.

There are finitely many consumers $\mathcal{H} = \{1, \dots, h, \dots, H\}$ and firms $\mathcal{J} = \{1, \dots, j, \dots, J\}$. Finitely many commodities $\mathcal{L} = \{1, \dots, l, \dots, L\}$ are traded at spot prices $p_s \in \mathbb{R}_+^L$ at every node $s \in \mathcal{S} = \{0, 1, \dots, S\}$. A production plan for firm j is a vector $y^j \in Y^j \subset \mathbb{R}^{L(S+1)}$. A consumption plan for consumer h is a vector $x^h \in \mathbb{R}_+^{L(S+1)}$.

At $t = 0$, shares of the J firms are also traded at prices $q \in \mathbb{R}_+^J$. For simplicity we assume that no other financial instruments are available for

⁸Excess supply of shares at a zero price is the translation, in our model, of under-subscription of a new emission.

trade. Saving and insurance needs of a consumer are reflected in her chosen portfolio of firms' shares, $\theta^h \in \mathbb{R}_+^J$ (no short sales).

Consumer h is described by her utility function : $u^h : \mathbb{R}_+^{L(S+1)} \rightarrow \mathbb{R}$, by her initial endowment of commodities, $w^h \in \mathbb{R}_+^{L(S+1)}$ and by her initial endowment of shares: $\delta^h \in [0, 1]^J$.

We make the following assumptions on the fundamentals of our economy:

For each consumer h , the utility function u^h is **(A.1)** continuously differentiable, **(A.2)** quasi-concave, **(A.3)** weakly monotone and **(A.4)** strictly monotone in good $l = 1$ in every date-event s ; **(A.5)** $w^h \in \mathbb{R}_{++}^{L(S+1)}$ and $\sum_{h \in \mathcal{H}} \delta^{hj} = 1, \forall j \in J$;

For each firm j , the production set $Y^j \subseteq \mathbb{R}^{L(S+1)}$ **(B.1)** is convex, **(B.2)** is closed, **(B.3)** satisfies free disposal: $-\mathbb{R}_+^{L(S+1)} \subset Y^j$; moreover, **(B.4)** $(\sum_{h \in \mathcal{H}} w^h + \sum_{j \in J} Y^j) \cap \mathbb{R}_+^{L(S+1)}$ is compact.

2.1 Consumer choices

For given prices (p, q) and given production choices of all firms, $y = (y^j)_{j \in J}$, the budget set of consumer h , $\mathbb{B}^h(p, q, y)$, is defined by the inequalities:

$$p_0 x_0^h + \sum_{j \in J} q^j \theta^{hj} \leq p_0 w_0^h + \sum_{j \in J} q^j \delta^{hj} + \sum_{j \in J} \theta^{hj} (p_0 y_0^j), \quad (2)$$

$$p_s x_s^h \leq p_s w_s^h + \sum_{j \in J} \theta^{hj} (p_s y_s^j) \text{ for all } s = 1, \dots, S. \quad (3)$$

The consumer chooses her portfolio and consumption bundle so as to maximize her utility on the budget set, leading to the first order conditions:

$$\frac{\partial u^h}{\partial x_{ls}^h} = \lambda_s^h p_{ls}, \text{ for all } s = 0, 1, \dots, S \quad (4)$$

$$\lambda_0^h q^j \geq \sum_s \lambda_s^h (p_s y_s^j) \text{ for all } j \in J \quad (5)$$

$$\theta^{hj} (\lambda_0^h q^j - \sum_s \lambda_s^h (p_s y_s^j)) = 0 \text{ for all } j \in J \quad (6)$$

Define $\bar{\lambda}_s^h = \frac{\lambda_s^h}{\lambda_0^h}$. From (3), $\bar{\lambda}_s^h$ is consumer h 's marginal rate of substitution between revenue in state s tomorrow and revenue today. Equation (5) can then be written :

$$q^j \geq \sum_s \bar{\lambda}_s^h (p_s y_s^j) \text{ for all } j \in J$$

with equality if $\theta^{jh} > 0$.

2.2 Price theories and the objective of the firm

Drèze (1974) and Grossman and Hart (1979) proposed criteria for the firm based on the idea that shareholders should not be able to identify a change of production plan and a set of transfers at $t = 0$ which make all of them better off.

Following Grossman and Hart, we assume that production plans are decided at $t = 0$ by the original shareholders⁹. A production plan y^j is (initial) *shareholder-efficient* if there does not exist a feasible change dy^j such that :

$$\sum_{h \in \mathcal{H}^j} \frac{du^h}{\lambda_0^h} > 0$$

where $\mathcal{H}^j := \{h \in \mathcal{H} : \delta^{hj} > 0\}$ is the set of initial shareholders of firm j .

To interpret this expression, consider the effect of a marginal change of production at node s , dy_s^j , on the utility of consumer $h \in \mathcal{H}^j$. Using the consumer budget constraint and first order conditions we have:

$$\frac{du^h}{\lambda_0^h} = \frac{\lambda_s^h}{\lambda_0^h} \theta^{hj} (p_s dy_s^j) + \frac{\lambda_0^h}{\lambda_0^h} (\delta^{hj} - \theta^{hj}) dq^{hj}$$

As in the standard GEI model, we assume that consumers have correct expectations of equilibrium prices, and plan their consumption and portfolio choices accordingly. When, as shareholders of firm j , they contemplate a proposed change in the firm's production plan, they consider the effect on the dividends received at $t = 1$, but also the capital gains or losses that the proposed change would induce through its effect on the shares' price at $t = 0$.

⁹Drèze assumes instead that production plans are decided by final shareholders, in order to elucidate the prospects for Constrained Pareto Optimality.

To evaluate this effect they need to form expectations on the way in which the price of the stock would be affected, dq^{hj} .

We call these expectations the consumer's *price theory*. In principle a price theory may be very complicated, for example it could be derived from a thorough knowledge of the structure of the economy and of the new equilibrium that would obtain if the production change were to be implemented. At the other extreme, it could be a purely subjective belief.

We retain the idea, introduced by Grossman and Hart, that "competitive price theories" can be fully characterized, for any given observed prices and production plans, (p, q, y) , by a vector of coefficients $\alpha \in R_{++}^{S+1}$; if the original shareholder h of firm j has a price theory α^{hj} , she evaluates the change in price following the proposed dy^j as:

$$dq^{hj} = \sum_s \bar{\alpha}_s^{hj} (p_s dy_s^j)$$

with $\bar{\alpha}_s^{hj} = \frac{\alpha_s^{hj}}{\alpha_0^{hj}}$.

The effect on her utility of a marginal change of production at node s is then calculated as:

$$\frac{du^h}{\lambda_0^h} = \bar{\lambda}_s^h \theta^{hj} (p_s dy_s^j) + (\delta^{hj} - \theta^{hj}) \bar{\alpha}_s^{hj} (p_s dy_s^j)$$

Thus, the coefficient

$$\beta_s^{hj} = \bar{\lambda}_s^h \theta^{hj} + (\delta^{hj} - \theta^{hj}) \bar{\alpha}_s^{hj}$$

is consumer h 's marginal valuation, at the proposed allocation, of an additional unit of firm j 's profit at node s .

Initial shareholder efficiency can thus be expressed by saying that there should not exist a change of production plan dy^j such that:

$$\sum_{h \in \mathcal{H}^j} \frac{du^h}{\lambda_0^h} = \sum_{h \in \mathcal{H}^j} \sum_s \beta_s^{hj} (p_s dy_s^j) > 0$$

Define $\beta_s^j = \sum_{h \in \mathcal{H}^j} \beta_s^{hj}$. Then the previous reasoning leads to:

Proposition: *Given prices (p, q) , production choices y and price theories α , the production plan chosen by firm j is shareholder-efficient if it solves the*

problem

$$\max_{y \in Y^j} \sum_s \beta_s^j (p_s y_s^j)$$

Remark 1. If h is buying shares of j in $t = 0$, and thinks that $\bar{\alpha}_s^{hj} \gg \bar{\lambda}_s^h$ she might be in favor of a plan that *reduces* profits in s , $p_s dy_s^j < 0$, because she expects a substantial drop in price today after the announcement of the new plan.

Remark 2. The Grossman and Hart (1979) hypothesis, unfortunately¹⁰ labelled "competitive price perceptions", introduces a particular price theory $\bar{\alpha}^{hj} = \bar{\lambda}^h$, so that:

$$dq^{jh} = \sum_s \bar{\lambda}_s^h (p_s dy_s^j).$$

Under this price theory, when each consumer assumes that the price exactly incorporates her own evaluation of the proposed production plan, the dividends and price terms in our definition of β_s^{hj} cancel out, leading to the Grossman-Hart criterion:

$$\begin{aligned} \beta_s^{hj} &= \bar{\lambda}_s^h \theta^{hj} + (\delta^{hj} - \theta^{hj}) \bar{\alpha}_s^h \\ &= \bar{\lambda}_s^h \delta^{hj} \end{aligned}$$

Remark 3. Even if all original shareholders of firm j use the same price theory, $\bar{\alpha}^{hj} = \bar{\alpha}^j$, the state price vector communicated to the firm need not coincide with $\bar{\alpha}^j$:

$$\beta_s^j = \sum_{h \in \mathcal{H}^j} \beta_s^{hj} = \sum_{h \in \mathcal{H}^j} \bar{\lambda}_s^h \theta^{hj} + \bar{\alpha}_s^j (1 - \sum_{h \in \mathcal{H}^j} \theta^{hj})$$

Indeed there may still remain differences in the evaluation of the dividend effect, as reflected in the expression for β^j . Two special cases are worth mentioning. If all original shareholders plan to sell completely their shares, $\theta^{hj} = 0$ for all $h \in \mathcal{H}^j$, all that matters for them is the price effect and we obtain $\beta^j = \bar{\alpha}^j$. If, on the other hand, the original shareholders only exchange shares between themselves so that original and final shareholders coincide, $\sum_{h \in \mathcal{H}^j} \theta^{hj} = 1$, we obtain

$$\beta_s^j = \sum_{h \in \mathcal{H}^j} \bar{\lambda}_s^h \theta^{hj}$$

¹⁰See the discussion in paragraph 1.2.2 of the introduction.

an analog of the criterion originally studied by Drèze (1974).

Remark 4. In a series of paper, Makowski (1983a,b, Makowski and Pepall 1985), and then Allen and Gale (1988, 1989) and Bisin, Gottardi and Ruta (2009), argued that, in a competitive market, shareholders of firm j should all use the following function to forecast the effect of a change in the production plan of the firm on its price:

$$\hat{q}^j(y^j) = \max_h \left[\sum_s \lambda_s^h y_s^j \right]$$

Makowski in particular was the first to prove that if all shareholders use this forecasting function, then they unanimously support the maximization of $\hat{q}^j(\cdot)$ as the objective of the firm. The function $\hat{q}^j(\cdot)$ does not fit our definition of an admissible price theory because it cannot be reduced to a vector of state prices. Starting from a production plan y^j , for any particular proposed change dy^j we could write

$$\bar{\alpha}^j(dy^j) = \bar{\lambda}^{h(dy^j)}$$

where $h(dy^j) = \arg \max_h [\sum_s \lambda_s^h (y_s^j + dy_s^j)]$: the λ 's of different individuals may be used to evaluate different production plans. The informational requirements are much more demanding than those embodied in our definition: for every possible deviation, one has to know the equilibrium λ 's of the individuals that would like to hold shares of the firm after *that* deviation.

3 Equilibrium

Let $\Delta_S := \{\alpha \in \mathbb{R}_+^{(S+1)} \mid \sum_s \alpha_s = 1\}$. Following the discussion in the previous section, we assume that every consumer h is characterized by price theory correspondences:

$$A^{hj} : \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^{J(S+1)} \times_j \mathbb{Y}^j \rightarrow \Delta_S$$

describing how individual h forms her price theories¹¹ from the observable variables (p, q, y) . We make the following continuity assumption:

¹¹Each consumer h actually needs a price theory only for the firms of which she is an original shareholder, but we omit this refinement.

(A.6) For every h and every j , A^{hj} is an upper hemi-continuous correspondence.

The *economy* is described by a collection:

$$\mathcal{E} = \{\mathcal{S}, \mathcal{L}, \mathcal{H}, \mathcal{J}, (u^h, w^h, ((\delta^{hj})_j, (A^{hj})_j)_h, (Y^j)_j)\}$$

satisfying assumptions $A(1) - A(6)$ and $B(1) - B(4)$.

An *equilibrium* for \mathcal{E} consists of spot prices p , stock prices q , price theories α , consumer state prices λ , consumption plans x , portfolios θ , production plans y , such that,

- (1) for each $j \in J$: $\sum_h \theta^{hj} = 1$;
- (2) for each $s = 0, 1, \dots, S$: $\sum_h (x_s^h - w_s^h) = \sum_{j \in J} y_s^j$;
- (3) for each h and each j , $\alpha^{hj} \in A^{hj}(p, q, y)$;
- (4) for each h :
 (x^h, θ^h) solves $\text{Max}_{(x^h, \theta^h)} \{u^h(x^h) \text{ s.t. } (x^h, \theta^h) \in \mathbb{B}^h(p, q, y)\}$, and λ^h is the vector of marginal utilities of revenue at all date-events;
- (5) for each $j \in J$: given its initial shareholders $\mathcal{H}^j := \{h \in H : \delta^{hj} > 0\}$,
 y^j solves $\text{Max}_{y^j \in Y^j} \sum_s \beta_s^j(\theta, \lambda, \alpha)(p_s y_s^j)$ where:
for all s ,

$$\beta_s^j(\theta, \lambda, \alpha) := \sum_{h \in \mathcal{H}^j} \beta_s^{hj}(\theta, \lambda, \alpha)$$

$$\text{with } \beta_s^{hj}(\theta, \lambda, \alpha) := \bar{\lambda}_s^h \theta^{hj} + (\delta^{hj} - \theta^{hj}) \bar{\alpha}_s^{hj}$$

Conditions (1) and (2) are the market clearing equations for assets and commodities. Condition (4) is consumers' optimisation. Condition (5) requires optimal behavior by firms given our criterion and the admissible price theories of initial shareholders ((3)).

4 Existence

Radner (1972) was the first to address the issue of existence in an economy with uncertainty and incomplete markets. He assumed that each firm chooses its production plan y to maximize a (continuous and strictly concave) firm specific utility function u^j defined over the vector of dividends at all date-events. In his formulation, the objective of the firm is exogenously given and not derived from the preferences of shareholders. Radner could not show the existence of an equilibrium, but only of a weaker notion, which he called *pseudo-equilibrium*, in which conditions (1) and (2) in the definition are weakened to:

(1') for each $j \in J$: $q^j(\sum_h \theta^{hj} - 1) = 0$ and $\sum_h \theta^{hj} \leq 1$;

(2') for each $s = 0, 1, \dots, S$: $\sum_h (x_s^h - w_s^h) = \sum_{j \in J} (\sum_h \theta^{hj}) y_s^j$;

that is, at a pseudo-equilibrium there may be free disposal of shares and downscaling of production when the price of a firm is zero.

Our model is not a special case of Radner's. The most important difference is that in our case the objective of the firm is derived from a notion of shareholders' optimality and is therefore endogenous: the β 's used by the firm to discount profits are functions of shareholders' consumption and portfolio choices. Still, we can prove that a pseudo-equilibrium exists in our model:

Theorem 1. *Under assumptions (A.1) – (A.6) and (B.1) – (B.4) there exists a pseudo-equilibrium for \mathcal{E} .*

Grossman and Hart (1979), with an objective of the firm derived, as in our case, from the preferences of the shareholders, were able to prove the existence of a full-fledged equilibrium.

The crucial step in their proof uses a property, which we call *consistency*, stating that, at a pseudo-equilibrium, for each firm j :

$$q^j \geq \sum_s \beta_s^j (p_s y_s^j).$$

This property, an easy consequence of the Grossman - Hart assumption that for every h $\alpha^h = \lambda^h$, plays an important role in going from a pseudo-equilibrium to an equilibrium. Indeed, when paired with the fact that

inaction is always possible, it implies that, whenever $q^j = 0$ at a pseudo-equilibrium, it must also be the case that $\sum_s \beta_s^j(p_s y_s^j) = 0$. A proportional rescaling of the production plan then remains optimal for the firm, and one can reinterpret a pseudo-equilibrium as an equilibrium with $y^{j'} = \sum_h \theta^{hj} y^j$ and $\theta^{hj'} = \frac{\theta^{hj}}{\sum_h \theta^{hj}}$.

As the following example illustrates, with our broader notion of admissible price theories it may well happen that at a pseudo-equilibrium the property of consistency is not verified: $q^j < \sum_s \beta_s^j(p_s y_s^j)$ for some firm j ($j = 3$ in the example).

Example 1. Consider an economy in which uncertainty is described by two possible states at $t = 1$, $\mathcal{S} = \{0, s, s'\}$. One commodity is traded in each state, $\mathcal{L} = \{1\}$.

There are three individuals, $\mathcal{H} = \{1, 2, 3\}$, with the same utility function

$$u = \log x_0 + \frac{1}{2} \log x_s + \frac{1}{2} \log x_{s'}$$

and endowments of goods: $w^1 = (\frac{11}{2}, 30, 3)$, $w^2 = (\frac{11}{2}, 3, 30)$, $w^3 = (\frac{3}{2}, 0, 0)$.

There are also three firms, $\mathcal{J} = 1, 2, 3$, with technologies

$$Y^1 = \{(y_0, 0, -\frac{3}{2}y_0) \mid -\frac{1}{2} \leq y_0 \leq 0\}$$

$$Y^2 = \{(y_0, -\frac{3}{2}y_0, 0) \mid -\frac{1}{2} \leq y_0 \leq 0\}$$

$$Y^3 = \{(y_0, -\frac{5}{6}(y_0)^{\frac{3}{4}}, -\frac{5}{6}(y_0)^{\frac{3}{4}}) \mid y_0 \leq 0\}$$

and initial ownership as follows: $\delta^1 = (\frac{1}{2}, \frac{1}{2}, \frac{27}{56})$, $\delta^2 = (\frac{1}{2}, \frac{1}{2}, \frac{27}{56})$, $\delta^3 = (0, 0, \frac{2}{56})$.

One can show that all the conditions in the definition of a pseudo-equilibrium are verified with:

- Prices: $q = (0, 0, 0)$.
- Price theories: $\alpha^1 = (1, \frac{2}{5}, \frac{2}{5})$, $\alpha^2 = (1, \frac{2}{5}, \frac{2}{5})$, $\alpha^3 = (1, \frac{1}{20}, \frac{1}{20})$.
- Firms' objectives: $\beta^1 = \bar{\lambda}^1 = (1, \frac{1}{12}, \frac{2}{3})$, $\beta^2 = \bar{\lambda}^2 = (1, \frac{2}{3}, \frac{1}{12})$,
 $\beta^3 = \bar{\lambda}^3 \theta^{33} + \delta^{13} \alpha^1 + \delta^{23} \alpha^2 + (\delta^{33} - \theta^{33}) \alpha^3 = (1, \frac{4}{5}, \frac{4}{5})$.
- Production plans: $y^1 = (-\frac{1}{2}, 0, \frac{3}{4})$, $y^2 = (-\frac{1}{2}, \frac{3}{4}, 0)$, $y^3 = (-1, \frac{5}{6}, \frac{5}{6})$.

- Shares: $\theta^1 = (1, 0, 0)$, $\theta^2 = (0, 1, 0)$, $\theta^3 = (0, 0, \frac{3}{4})$.
- Consumption plans: $x^1 = (5, 30, \frac{15}{4})$, $x^2 = (5, \frac{15}{4}, 30)$, $x^3 = (\frac{3}{4}, \frac{5}{8}, \frac{5}{8})$.

The example shows that at a pseudo equilibrium the actual, rescaled production plan need not be optimal for the firm, so that a full equilibrium cannot be attained by rescaling production and shareholdings. Firm 3 plans $y_0^3 = -1$, but, given $\theta^{33} = \frac{3}{4}$, its actual input is only $\theta^{33}y_0^3 = -\frac{3}{4}$. With that input, efficient production is $y_s^{3'} = y_{s'}^3 = \frac{5}{6}(\frac{3}{4})^{\frac{3}{4}} \approx 0.67$, higher than the rescaled output $\theta^{33}y_s^3 = \theta^{33}y_{s'}^3 = \frac{3}{4} \cdot \frac{3}{4} \approx 0.62$. \square

4.1 Constraints on investment

As we argued in the previous section, under the stated assumptions we cannot prove the existence of an equilibrium for our economy.

The problem arises when the firm plans an investment but its price is equal to zero and there is an excess supply of its shares, so that the required financing cannot be collected from the shareholders.

In the example, the objective of firm 3 (see the expression for β^3) takes into account the marginal evaluations of individuals 1 and 2, who plan to sell their shares and anticipate a capital gain. Individual 3 has a different price theory and plans to buy shares, but he is not ready to subscribe the total planned investment.

The non existence of a full fledged equilibrium reflects the fact that with incomplete markets, even under the strong hypothesis of perfect foresight, publicly observable market data need not provide enough information to bring consistency in individuals' evaluations of possible capital gains and losses. When this is the case, the aggregation of shareholders marginal evaluations may lead the firm to the choice of a production that cannot be fully financed.

This observation suggests modifying the notion of equilibrium by adding to the maximization problem of the firm a *constraint on fund raising* which becomes binding exactly in this case. We illustrate the idea in the context of the previous example.

Example 1 (continued). At the pseudo-equilibrium calculated above, firm 3 plans to use input $y_0^3 = -1$, but, given $\theta^{33} = \frac{3}{4}$, its actual input is only $\theta^{33}y_0^3 = -\frac{3}{4}$.

Suppose now that, when the price of a firm is zero, a quantity constraint comes into action, limiting the investment of the firm. That is, let us introduce parameters $k = (k^j < 0)_{j \in J}$ and impose that, if $q^j = 0$, the firm production plan must satisfy:

$$y_0^j \geq k^j$$

The parameters k are equilibrating variables, determined at equilibrium to assure consistency and market clearing.

A constrained equilibrium is obtained with $k^3 = -\frac{3}{4}$ (and $k^1 = k^2 < -\frac{1}{2}$, non binding).

From the first order conditions of Mr. 3, at $q^3 = 0$, $y^3 = (-\frac{3}{4}, \frac{5}{6}(\frac{3}{4})^{\frac{3}{4}}, \frac{5}{6}(\frac{3}{4})^{\frac{3}{4}})$, he chooses $\theta^{33} = 1$.

The new value of $\beta_s^3 = \beta_{s'}^3$ is

$$\beta_s^3 = \beta_{s'}^3 = \bar{\lambda}^3 \theta^{33} + \delta^{13} \alpha^1 + \delta^{23} \alpha^2 + (\delta^{33} - \theta^{33}) \alpha^3 \approx 0.89 > \frac{4}{5}$$

Given the new β^3 , at the solution of the firm 3's problem the constraint is binding and the firm is optimizing at $-y_0^3 = \frac{3}{4}$. \square

The idea illustrated in the example can be generalized. Let us define an *investment-constrained equilibrium* for \mathcal{E} as a collection of spot prices p , stock prices q , price theories α , consumer state prices λ , consumption plans x , portfolios θ , production plans y , and constraints $k \in \mathbb{R}_-^J$ such that,

- (1) for each $j \in J$: $\sum_h \theta^{hj} = 1$;
- (2) for each $s = 0, 1, \dots, S$: $\sum_h (x_s^h - w_s^h) = \sum_{j \in J} y_s^j$;
- (3) for each h and each j , $\alpha^{hj} \in A^{hj}(p, q, y)$;
- (4) for each h :
 (x^h, θ^h) solves $\text{Max}_{(x^h, \theta^h)} \{u^h(x^h) \text{ s.t. } (x^h, \theta^h) \in \mathbb{B}^h(p, q, y)\}$, and λ^h is the vector of marginal utilities of revenue at all date-events;
- (5) for each $j \in J$: given its initial shareholders $\mathcal{H}^j := \{h \in H : \delta^{hj} > 0\}$,
 y^j solves $\text{Max}_{y^j \in \mathbb{Y}^j(p, k^j)} \sum_s \beta_s^j(\theta, \lambda, \alpha)(p_s y_s^j)$ where:

$$\mathbb{Y}^j(p, k^j) = \{y^j \in Y^j \mid p_0 y_0^j \geq k^j\}$$

and, for all s ,

$$\beta_s^j(\theta, \lambda, \alpha) := \sum_{h \in \mathcal{H}^j} \beta_s^{hj}(\theta, \lambda, \alpha)$$

$$\text{with } \beta_s^{hj}(\theta, \lambda, \alpha) := \bar{\lambda}_s^h \theta^{hj} + (\delta^{hj} - \theta^{hj}) \bar{\alpha}_s^{hj}$$

(6) if $q^j > 0$, then $p_0 y_0^j > k^j$.

To prove the existence of an investment-constrained equilibrium we need to be sure that the feasible set of every firm j , \mathbb{Y}^j , is continuous in the relevant variables. To this end, we make the following two additional assumptions¹²:

(A.0) For each consumer h , u^h is strictly monotone in x_{0l} for every $l \in \mathcal{L}$.

(B.0) For each firm j , $Y^j \subset \mathbb{R}_-^L \times \mathbb{R}_+^{LS}$, and for all $y^j \in Y^j$, $y_0^j = 0$ implies $y^j = 0$.

Theorem 2. *Under assumptions (A.0)–(A.6) and (B.0)–(B.4), there exists an investment-constrained equilibrium for \mathcal{E} .*

4.2 No arbitrage and consistency

In our definition of the economy we take the price theories of individuals as given and do not impose any restriction on them except a form of continuity. As we argued in the introduction, even under strong assumptions of rationality and perfect foresight of equilibrium prices, there is no reason, when markets are incomplete, to assume that individuals form correct out-of-equilibrium expectations of share-price changes when they consider possible deviations from the equilibrium production plans. Still, one may want to

¹²An analogous problem arises in the theory of equilibria with price rigidities and quantity constraints on net sales. In the latter case, existence of equilibrium requires a special assumption (like initial endowments in the interior of the consumption sets, or strictly positive prices for all commodities). The same is true here. But firms are not assumed to hold initial endowments. So, we fall back on strictly positive prices (implied by A.0), and we need to add strictly positive investments (i.e. a production plan with inputs only and no initial outputs, (B.0)). There clearly remains scope for further investigation of this more general approach.

investigate whether *some* restrictions on admissible price theories could be deduced from knowledge of the equilibrium values of prices and production plans (p, q, y) .

An important idea coming from the theory of finance is that the absence of arbitrage implies that the prices of traded assets can be explained on the basis of underlying "state prices" reflecting the market evaluation of revenue in the possible states of the world. An implication of this idea is that, based only on the observation of equilibrium prices and payoffs, one can put restrictions on the estimation of the market evaluation of revenue in the different states.

More precisely, from the fundamental theorem of asset pricing (Ross (1978)), we know that, for given spot prices and production plans, (p, y) , the stock prices q satisfy *no - arbitrage*, $q \in NA(p, y)$, if and only if there exists state prices $\alpha \in \mathbb{R}_{++}^{S+1}$ such that

$$\alpha_0(q^j - p_0 y_0^k) = \sum_{s>0} \alpha_s(p_s y_s^k) \text{ for all } k \in J. \quad (7)$$

When markets are complete there is only one solution to this equation, and the market evaluation of revenue in the possible states can be perfectly inferred from the observation of equilibrium prices and production plans. When markets are incomplete the solution is not unique, but the set of admissible α 's is nevertheless restricted.

One could then think that each individual should use as her price theories vectors of state prices compatible with the observable variables and the assumption of no arbitrage.

Indeed, it is easy to show that, if at a pseudo-equilibrium equation (7) is satisfied by the price theories of every individual, the property of consistency holds and the pseudo-equilibrium can be turned into a full fledged equilibrium.

In our model though, there need not be *any* vector of state prices satisfying equation (7) at the pseudo-equilibrium, due to the assumption of no short selling¹³ ($\theta^{hj} \geq 0$ for all h and j).

When short sales are not allowed, a weaker necessary condition holds at a pseudo-equilibrium, no-unlimited-arbitrage, as characterized in Theorem 7.3.2 of Leroy and Werner (2001, p.68): for given spot prices and

¹³This can be confirmed by adapting to our context Example 4.4.1, p. 36, of Leroy and Werner (2001).

production plans, (p, y) , the stock prices q satisfy *no - unlimited arbitrage*, $q \in NUA(p, y)$, if and only if there exists $\alpha \in \mathbb{R}_{++}^{S+1}$ such that

$$\alpha_0(q^j - p_0 y_0^k) \geq \sum_{s>0} \alpha_s(p_s y_s^k) \text{ for all } k \in J. \quad (8)$$

We may then formalize a restriction on admissible price theories as follows.

For given (p, q, y) with $q \in NUA(p, y)$, let $\Omega(p, q, y)$ be the set of vectors $\alpha \in \mathbb{R}_{++}^{S+1}$ satisfying condition (8) above. Let $\Delta_S := \{\alpha^h \in \mathbb{R}_+^{(S+1)} \mid \sum_s \alpha_s^h = 1\}$ and assume that every consumer h is characterized by price theory correspondences

$$A^{hj} : \mathbb{R}_+^{L(S+1)} \times \mathbb{R}_+^{J(S+1)} \times_j \mathbb{Y}^j \rightarrow \Delta_S$$

satisfying¹⁴

(A.6') For every h and every j , A^{hj} is an upper hemi-continuous correspondence such that $A^{hj}(p, q, y) \in \Omega(p, q, y) \cap \Delta_S$ for any (p, q, y) such that $q \in NUA(p, y)$.

This is the correct expression, in our model, of the idea that no-arbitrage arguments should discipline the price theories of individuals.

Unfortunately, as illustrated in Example 1, imposing this discipline is not enough to obtain consistency. Indeed, in the example, the price theories of the three consumers do satisfy assumption (A.6').

The weak form of no arbitrage compatible with our model allows us to prove the existence of equilibrium under the additional assumption of common price theories among the initial shareholders of any given firm¹⁵:

(A.7) For all j , if $h, h' \in \mathcal{H}_j$, then $A^{hj} = A^{h'j}$.

¹⁴Correspondences satisfying (A.6') exist, for example one could use the map that associates to any (p, q, y) the (normalized) marginal utilities of revenue, λ^h 's, of a given consumer h .

¹⁵Assumptions (A.6') and (A.7) do not guarantee the existence of an equilibrium in a model with more than two periods. Even if two shareholders agree on the anticipated variation of the firm price at a node s_t in period $t > 0$, they will in general use different $\lambda_{s_t}^h$'s to discount that variation back in terms of $t = 0$ revenue.

Theorem 3. *Under assumptions (A.1) – (A.5), (B.1) – (B.4), (A.6') and (A.7), there exists an equilibrium for \mathcal{E} .*

The notion of equilibrium and its relationship with alternative notions are illustrated in the following example¹⁶, adapted from the one in Duffie and Shafer (1986) and Dierker and Dierker (2007).

Example 2. Consider an economy in which uncertainty is described by two possible states at $t = 1$, $\mathcal{S} = \{0, s, s'\}$. One commodity is traded in each state, $\mathcal{L} = \{1\}$. There are two consumers, $\mathcal{H} = \{1, 2\}$, with utility functions

$$u^1 = x_0 + \frac{1}{6}(2\log x_s + \log x_{s'})$$

$$u^2 = x_0 + \frac{1}{3}(\log x_s + 2\log x_{s'})$$

and endowments: $w^1 = (1, 0, 0)$, $w^2 = (1, 1, 1)$. There is one firm, $\mathcal{J} = 1$, owned by consumer 2, $\delta^1 = 0$, $\delta^2 = 1$, with a technology such that the cost to produce $(y_s, y_{s'}) = (Y, Z)$ tomorrow is $y_0 = -(Y^2 + Z^2)$.

Given a criterion $\beta = (1, \beta_s, \beta_{s'})$, the firm chooses (Y, Z) to maximize

$$\Pi = -(Y^2 + Z^2) + \beta_s Y + \beta_{s'} Z,$$

leading to the choice of a production plan $(y_0, Y, Z) = (-\frac{\beta_s^2 + \beta_{s'}^2}{4}, \frac{\beta_s}{2}, \frac{\beta_{s'}}{2})$.

At a Grossman-Hart equilibrium, Mr. 2 sells the firm to Mr.1, $\theta^2 = 0$, so that $\beta_s = \lambda_s^2 = \frac{1}{3}$ and $\beta_{s'} = \lambda_{s'}^2 = \frac{2}{3}$, and the chosen production plan is $(y_0^{GH}, Y^{GH}, Z^{GH}) = (-\frac{5}{36}, \frac{1}{6}, \frac{1}{3})$.

To calculate our proposed solution, notice that, under A.7, a price theory α is admissible if at the observed (q, Y, Z) it satisfies the NUA inequality

$$q + (Y^2 + Z^2) \geq \alpha_s Y + \alpha_{s'} Z. \quad (9)$$

If the initial owner, consumer 2, has a price theory α and plans to sell the firm, then $\beta = \alpha$ and the chosen production plan is $y^\alpha = (-\frac{\alpha_s^2 + \alpha_{s'}^2}{4}, \frac{\alpha_s}{2}, \frac{\alpha_{s'}}{2})$. To have $\theta^1 = 1$, the firm price must then be

$$q = \frac{1}{2} - \frac{\alpha_s^2 + \alpha_{s'}^2}{4} \quad (10)$$

¹⁶In the example we do not specify the individual maps A^{hj} 's. Implicitly we are assuming that consumers may have *any* price theory satisfying the NUA admissibility condition. The example thus illustrates the potential multiplicity of equilibria.

Combining equations (9) and (10), we see that there exists an equilibrium for any α such that

$$\alpha_s^2 + \alpha_{s'}^2 \leq 1$$

At all of these equilibria $\theta^1 = 1$ and $\theta^2 = 0$, but different values of α lead to different production choices.

Among the admissible price theories, consumer 2 may use the true marginal evaluation of the buyer, $\alpha = \lambda^1$. leading to a Drèze equilibrium, with $Y^D = \sqrt{\frac{1}{6}}$, $Z^D = \sqrt{\frac{1}{12}}$. Notice though that, if the only observables are (q, Y, Z) , *all* equilibria sustained by α such that $\alpha_s^2 + \alpha_{s'}^2 \leq 1$ are equally admissible. \square

5 Concluding comments

The main contributions of this paper have been: (i) to introduce a reasonable concept of shareholder efficiency for a standard production economy with incomplete markets, a concept that reveals the role of anticipations about the impact of future state-dependent profits on current stock prices; (ii) to bring out the potential role of financial constraints when these anticipations remain idiosyncratic and largely unconstrained, a role embodied in the realistic concept of investment-constrained equilibrium; (iii) to prove a general existence theorem for that equilibrium concept; and (iv) to record some implications of the property that anticipations be consistent with no-unlimited-arbitrage.

There remains scope for: (v) exploring further reasonable constraints on the anticipations, and (vi) extending our analysis to more realistic models, including decisions resulting from the control principle with endogenous boards of directors, also in a Temporary General Equilibrium framework.

Less immediately, there is scope for integrating more ambitiously the general equilibrium approach with finance theory. The fact that constrained optima may fail to be sustained by state prices (see the discussion in section 1.2.2) is an important departure from standard modeling in finance. The leeway admitted by equilibrium for anticipations of price derivatives (for our price theories) is a warning of potential proliferation of assets created on the basis of erratic anticipations. It is also a warning of potential deviations linked to stock options. There is an undeniable flavour of current relevance to these developments, which deserve research priority.

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Appendix

We first define a modified notion of pseudo-equilibrium, which we use as a tool in the proof of Theorem 3. As an intermediate step we prove a simple but crucial Consistency Lemma. The proofs of Theorems 1 and 2 follow the main steps in the proof of Theorem 3.

A1. Modified Pseudo-Equilibrium

Differently from the definition of pseudo-equilibrium, in a modified pseudo-equilibrium consumers are aware of possibility of free disposal of shares; they fully anticipate its occurrence, and revise their marginal valuation of an additional unit of revenue in a given state accordingly. We model this by introducing rescaling factors τ^j 's in the definition. The concept is not meant to have any descriptive appeal. We will use it only in an intermediate step in the proof of Theorem 1.

A *modified pseudo-equilibrium* for \mathcal{E} consists of spot prices p , stock prices q , price theories α , consumer state prices λ , consumption plans x , portfolios θ , production plans y , and anticipated *re-scaling factors* τ such that

- (1) for each $j \in J$: $q^j(\sum_h \theta^{hj} - 1) = 0$ and $\sum_h \theta^{hj} \leq 1$;
- (2) for each $s = 0, 1, \dots, S$: $\sum_h (x_s^h - w_s^h) = \sum_{j \in J} (\sum_h \theta^{hj}) y_s^j$;
- (3) for all h and j , $\alpha^{hj} \in A^{hj}(p, q, y)$;
- (4) for each h : (x^h, θ^h) solves $\text{Max}_{(x^h, \theta^h)} \{u^h(x^h) \text{ s.t. } (x^h, \theta^h) \in \mathbb{B}^h(p, q, y)\}$, and λ^h is the vector of marginal utilities of revenue at all date-events;
- (5) for each $j \in J$: given its initial shareholders $\mathcal{H}^j := \{h \in H : \delta^{hj} > 0\}$, y^j solves $\text{Max}_{y^j \in Y^j} \sum_s \beta_s^j(\theta, \lambda, \alpha, \tau)(p_s y_s^j)$ where for all s ,

$$\beta_s^j(\theta, \lambda, \alpha, \tau) = \sum_{h \in \mathcal{H}^j} \beta_s^{hj}(\theta, \lambda, \alpha, \tau)$$

$$\text{with } \beta_s^{hj}(\theta, \lambda, \alpha, \tau) := \tau^j \theta^{hj} \bar{\lambda}_s^h + (\delta^{hj} - \tau^j \theta^{hj}) \bar{\alpha}_s^{hj} \quad (11)$$

- (6) for each $j \in J$:

$$\tau^j = \frac{1}{\sum_h \theta^{hj}}.$$

In (5), the definition of the objective of the firm takes into account that free disposal of shares may alter the actual share of a firm held by a consumer at the pseudo-equilibrium. Condition (6) defines the rescaling factors τ^j in terms of the actual number of shares held by consumers.

A2. Consistency Lemma

Lemma 1 *At a modified pseudo-equilibrium, if any of the following conditions hold:*

i) for all h and all j , $\alpha^{hj} = \lambda^h$.

ii) if $h, h' \in \mathcal{H}_j$, then $\alpha^{hj} = \alpha^{h'j} = \alpha^j$, and α^j satisfies the NUA inequalities

$$\alpha_0^j(q^k - p_0 y_0^k) \geq \sum_{s>0} \alpha_s^j(p_s y_s^k) \text{ for all } k \in J.$$

iii) for all h and all j , α^{hj} satisfies the NA equations

$$\alpha_0^{hj}(q^k - p_0 y_0^k) = \sum_{s>0} \alpha_s^{hj}(p_s y_s^k) \text{ for all } k \in J.$$

then:

$$q^j \geq \sum_s \beta_s^j(p_s y_s^j) \text{ for all } j \in J.$$

Proof: At a modified pseudo-equilibrium, for all h , all j and all s

$$\beta_s^{hj} = \frac{\lambda_s^h}{\lambda_0^h} \tau^j \theta^{hj} + (\delta^{hj} - \tau^j \theta^{hj}) \frac{\alpha_s^{hj}}{\alpha_0^{hj}}.$$

Fixing j , multiply each β_s^{hj} by $(p_s y_s^j)$ and sum across $h \in \mathcal{H}^j$ and $s > 0$,

$$\sum_{h \in \mathcal{H}^j} \sum_{s>0} \beta_s^{hj}(p_s y_s^j) = \tau^j \sum_{h \in \mathcal{H}^j} \theta^{hj} \sum_{s>0} \frac{\lambda_s^h}{\lambda_0^h} (p_s y_s^j) + \sum_{h \in \mathcal{H}^j} [\delta^{hj} - \tau^j \theta^{hj}] \sum_{s>0} \frac{\alpha_s^{hj}}{\alpha_0^{hj}} (p_s y_s^j) \quad (12)$$

From this expression, if condition *i)* holds, we can use the first order conditions of the consumer's problem to obtain:

$$\begin{aligned} \sum_{h \in \mathcal{H}^j} \sum_{s>0} \beta_s^{hj}(p_s y_s^j) &= \sum_{h \in \mathcal{H}^j} \delta^{hj} \sum_{s>0} \frac{\lambda_s^h}{\lambda_0^h} (p_s y_s^j) \\ &\leq \sum_{h \in \mathcal{H}^j} \delta^{hj} (q^j - p_0 y_0^j) = (q^j - p_0 y_0^j). \end{aligned}$$

Consider now condition *ii*), then:

$$\begin{aligned}
\sum_{h \in \mathcal{H}^j} \sum_{s > 0} \beta_s^{hj} (p_s y_s^j) &= \tau^j \sum_{h \in \mathcal{H}^j} \theta^{hj} \sum_{s > 0} \frac{\lambda_s^h}{\lambda_0^h} (p_s y_s^j) \\
&+ \sum_{h \in \mathcal{H}^j} [\delta^{hj} - \tau^j \theta^{hj}] \sum_{s > 0} \frac{\alpha_s^j}{\alpha_0^j} (p_s y_s^j) \\
&\leq \sum_{h \in \mathcal{H}^j} [\tau^j \theta^{hj} + \delta^{hj} - \tau^j \theta^{hj}] (q^j - p_0 y_0^j) = \\
&= (q^j - p_0 y_0^j),
\end{aligned} \tag{13}$$

using again the first order conditions of the consumer's problem and the fact that α satisfies the NUA inequalities.

Finally, if *iii*) is verified, for all h, j and k , $\sum_{s > 0} \frac{\alpha_s^{hj}}{\alpha_0^{hj}} (p_s y_s^k) = (q^k - p_0 y_0^k)$. Using again the consumers' first order conditions we obtain:

$$\sum_{h \in \mathcal{H}^j} \sum_{s > 0} \beta_s^{hj} (p_s y_s^j) \leq (q^j - p_0 y_0^j) \sum_{h \in \mathcal{H}^j} [\tau^j \theta^{hj} + \delta^{hj} - \tau^j \theta^{hj}] = (q^j - p_0 y_0^j)$$

□

A3. Proof of Theorem 3

The proof is divided in three parts. In PART I we show that a modified pseudo equilibrium exists for an economy compactified by imposing some artificial bounds. In PART II we show that, starting from a modified pseudo-equilibrium, using (A.6'), (A.7) and the Consistency Lemma, we can construct an equilibrium for the compactified economy. Finally, in PART III we show that the artificial bounds can be removed.

PART I.

We normalize prices as follows: $(p_0, q) \in \Delta_{L+J-1}$, $p_s \in \Delta_{L-1}$ for all $s \neq 0$. The space of prices is thus $\mathbb{Q} = \Delta_{L+J-1} \times \prod_{s \geq 1} \Delta_{L-1}$. Moreover we take $\lambda^h \in \Delta_S$ and $\alpha^{hj} \in \Delta_S$, for all $h \in H, j \in J$.

Next, we impose some artificial bounds on choice spaces.

Define

$$\mathbb{F} := \left\{ (x, y) \in (\mathbb{R}^{L(S+1)})^H \times \left(\prod_{j \in \mathcal{J}} \mathbb{Y}^j \right) \mid \sum_{h \in \mathcal{H}} (x^h - w^h) - \sum_{j \in \mathcal{J}} y^j \leq 0 \right\}$$

as the set of feasible allocations of commodities. Then, according to Assumption (B.4), there exists a positive real number \bar{m} , such that, for any $m \geq \bar{m}$, $\mathbb{X}_m^h := \mathbb{R}_+^{L(S+1)} \cap M$ contains $\text{proj}_{\mathbb{R}_+^{L(S+1)}} \mathbb{F}$ in its interior for all $h \in \mathcal{H}$, and $\mathbb{Y}_m^j := \mathbb{Y}^j \cap M$ contains $\text{proj}_{\mathbb{Y}^j} \mathbb{F}$ in its interior for all $j \in \mathcal{J}$, where M defines the n -dimensional hyper-cube $[-m, m]^n$ with $n := (L(S+1)H + L(S+1)J)$.

We impose an artificial lower bound on portfolios $b^{hj}(\epsilon) := \epsilon \delta^{hj}$ to avoid $\sum_h \theta_s^{hj} = 0$ at any s . In Lemma 2 we show that for any m big enough we can always choose $\epsilon(m)$ small enough that $\mathbb{B}^h(p, q, y) \neq \emptyset$ on the relevant domain. Let $\Theta_m^h = [b^{hj}(\epsilon(m)), m]^{\mathcal{J}}$ denote the set of constrained portfolios.

Lemma 2 *Let $\mu = \min \{w_{\ell s}^h \mid \ell \in \mathcal{L}, h \in \mathcal{H} \text{ and } s \in \mathcal{S}\}$ and $Jm > \mu$. If $\epsilon(m) = (\frac{\mu}{Jm})^2$, then for any arbitrary $(p, q) \in \mathbb{Q}$ and $y \in \prod_{j \in \mathcal{J}} \mathbb{Y}_m^j$, consumer h 's budget set is non - empty, i.e. $\mathbb{B}^h(p, q, y) \neq \emptyset$.*

Proof: We prove the result by defining, for any m big enough, an $\epsilon(m)$ and a feasible portfolio such that the budget constraint is satisfied at any prices and production plans.

Let $0 < \eta < 1$ and $\underline{\theta}^h = \eta \delta^h$.

At this portfolio, the income of consumer h at node $s = 0$ is

$$\sum_{j \in \mathcal{J}} q^j \delta^{hj} + \sum_{\ell \in \mathcal{L}} p_{\ell 0} \left(w_{\ell 0}^h + \sum_{j \in \mathcal{J}} \eta \delta^{hj} y_{\ell 0}^j \right),$$

and, at a node $s \neq 0$,

$$\sum_{\ell \in \mathcal{L}} p_{\ell s} \left(w_{\ell s}^h + \sum_{j \in \mathcal{J}} \eta \delta^{hj} y_{\ell s}^j \right).$$

Each of the terms inside the parentheses is bounded below by $\mu - Jm\eta$ because $-m \leq y_{\ell s}^j$ for all s, ℓ and j . If $\eta < \frac{\mu}{Jm}$, the lower bound is positive.

If m is big enough, $\frac{\mu}{Jm} < 1$, and the feasibility of $\underline{\theta}^h$ is guaranteed by taking $\epsilon(m) = (\frac{\mu}{Jm})^2$. \square

Having bounded portfolio holdings away from zero and from above, we can limit the range of the scaling factors $\tau^j = \frac{1}{\sum_h \theta^{hj}}$ to a compact set $\mathbb{T} \subset \mathbb{R}_+$.

Call the *truncated economy* \mathcal{E}_m .

Proposition 3 *Assume (A.1) – (A.6) and (B.1) – (B.4). Then there exists a modified pseudo-equilibrium for \mathcal{E}_m .*

Proof of Proposition 3:

Step 1: Defining the fixed point correspondence.

Let $\mathbb{Z} = \mathbb{Q} \times \prod_{\mathcal{H}\mathcal{J}} \Delta_S \times \prod_{\mathcal{H}} \Delta_S \times \prod_{h \in \mathcal{H}} \mathbb{X}_m^h \times \prod_{h \in \mathcal{H}} \Theta_m^h \times \prod_{j \in \mathcal{J}} \mathbb{Y}_m^j \times \prod_{\mathcal{J}} \mathbb{T}$ and $z = (p, q, \alpha, \lambda, x, \theta, y, \tau)$. The pseudo-equilibria correspondence is defined as

$$\begin{aligned} \mathcal{G} : \mathbb{Z} &\rightrightarrows \mathbb{Z} \\ z &\mapsto ((\mathcal{C}^h(z))_h, (\mathcal{P}^j(z))_j, \mathcal{M}(z), (A^{hj}(z))_{hj}, (\Lambda^h(z))_h, (\mathcal{T}^j(z))_j) \end{aligned} \quad (14)$$

where $A^{hj} : \mathbb{Q} \times \prod_{j \in \mathcal{J}} \mathbb{Y}_m^j \rightarrow \Delta_S$ is the upper hemi-continuous correspondence describing h 's price theory, $\mathcal{T}^j : \prod_{h \in \mathcal{H}} \Theta_m^h \rightarrow \mathbb{T}$ is the continuous function defining firm j 's scaling factors, as in point (6) of the definition of modified pseudo-equilibrium; and the correspondences $\{(\mathcal{C}^h)_h, (\mathcal{P}^j)_j, \mathcal{M}, (\Lambda^h)_h\}$ are respectively, consumers' compensated demand correspondences, firms' choice correspondences, and the market auctioneers' correspondences, that we define as follows.

For any $h \in H$, define the consumer's compensated demand correspondence as

$$\begin{aligned} \mathcal{C}^h : \mathbb{Z} &\rightrightarrows \mathbb{X}_m^h \times \Theta_m^h \\ z &\mapsto \left\{ \begin{array}{l} (x^h, \theta^h) \in \mathbb{X}_m^h \times \Theta_m^h \mid \lambda^h \cdot G^h(p, q, y, x^h, \theta^h) \leq 0; \\ u^h(x^h) \geq u^h(x) \text{ for all } (x, \theta) \in \mathbb{X}_m^h \times \Theta_m^h \\ \text{s.t. } \lambda^h \cdot G^h(p, q, y, x, \theta) < 0 \end{array} \right\} \end{aligned} \quad (15)$$

where G^h is the column vector whose elements are $(p_0(x_0^h - w_0^h) - \sum_j (\delta^{hj} - \theta^{hj})q^j - \sum_j \theta^{hj}(p_0 y_0^j))$ for the first component and $(p_s(x_s^h - w_s^h) - \sum_j \theta^{hj}(p_s y_s^j))$ for the remaining S .

For any $j \in \mathcal{J}$, let $\underline{\beta}^j$ the mapping defined from $\prod_{h \in \mathcal{H}} \Theta_m^h \times \prod_{\mathcal{H}\mathcal{J}} \Delta_S \times \prod_{\mathcal{H}} \Delta_S \times \prod_j \mathbb{T}^j$ to $\mathbb{R}^{(S+1)}$ by

$$\begin{aligned} \underline{\beta}_0^j(\theta, \lambda, \alpha, \tau) &= \prod_h \lambda_0^h \alpha_0^{hj} \text{ and, for } s \neq 0, \\ \underline{\beta}_s^j(\theta, \lambda, \alpha, \tau) &= \sum_{h \in \mathcal{H}_0^j} \left(\alpha_0^{hj} \lambda_s^h \tau^j \theta^{hj} + \lambda_s^h (\delta^{hj} - \tau^j \theta^{hj}) \alpha_s^{hj} \right) \prod_{h' \neq h} \lambda_0^{h'} \alpha_0^{h'j} \end{aligned}$$

Define the set $\mathbb{S}_m^j = \underline{\beta}^j(\prod_{h \in \mathcal{H}} \Theta_m^h \times \prod_{\mathcal{H}\mathcal{J}} \Delta_S \times \prod_{\mathcal{H}} \Delta_S \times \prod_{\mathcal{J}} \mathbb{T})$. Then, let V^j be the real valued mapping defined on $\mathbb{S}_m^j \times \text{proj}_{\mathbb{R}_+^{L(S+1)}}(\mathbb{Q}) \times \mathbb{Y}_m^j$ by $V^j(\underline{\beta}^j; p, y^j) = \sum_{s \in \mathcal{S}} \underline{\beta}_s^j(\theta, \lambda, \alpha, \tau)(p_s y_s^j)$.

Then, for any $j \in J$,

$$\begin{aligned} \mathcal{P}^j : \mathbb{Z} &\rightrightarrows \mathbb{Y}_m^j \\ z &\mapsto \text{Argmax} \{ V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau); p, y^j) \mid y^j \in \mathbb{Y}_m^j \}. \end{aligned} \quad (16)$$

is the firm's production correspondence.

For any $h \in H$, let

$$\begin{aligned} \Lambda^h : \mathbb{Z} &\rightrightarrows \Delta_S \\ z &\mapsto \text{Argmax} \{ \lambda^h \cdot G^h(p, q, x, \theta, y) \mid \lambda^h \in \Delta_S \}. \end{aligned} \quad (17)$$

be consumer h 's Lagrange multipliers correspondence.

Finally, let the auctioneer's choice correspondence be

$$\begin{aligned} \mathcal{M} : \mathbb{Z} &\rightrightarrows \mathbb{Q} \\ z &\mapsto \mathcal{M}(z) \end{aligned} \quad (18)$$

where

$$\mathcal{M}_0(z) = \text{Argmax}_{(p_0, q) \in \Delta_{L+J-1}} \left\{ p_0 \left(\sum_h (x_0^h - w_0^h - \sum_j \theta^{hj} y_0^j) \right) + \sum_j q^j \left(\sum_h \theta^{hj} - 1 \right) \right\}$$

and, for $s \neq 0$,

$$\mathcal{M}_s(z) = \text{Argmax}_{p_s \in \Delta_{L-1}} \left\{ p_s \left(\sum_h (x_s^h - w_s^h - \sum_j \theta^{hj} y_s^j) \right) \right\}$$

The correspondence \mathcal{G} embodies all equilibrium correspondences for consumers, producers, and the auctioneer, respectively. The set \mathbb{Z} is compact and convex by construction and Assumptions (B.1) and (B.2). The set \mathbb{S} is compact as well. We need to show that the correspondence \mathcal{G} satisfies the hypotheses of Kakutani's Fixed-Point Theorem, so that a fixed point z_m (we omit the subscript m from variables, unless specified otherwise) exists.

Lemma 4 *The compensated demand correspondences $(\mathcal{C}^h)_h$, the supply correspondences $(\mathcal{P}^j)_j$ and the market agents' correspondences \mathcal{M} and $(\Lambda^h)_h$, are non empty and convex valued, and upper hemi-continuous.*

Proof: The proofs for $(\mathcal{C}^h)_h$, \mathcal{M} and $(\Lambda^h)_h$ are standard; we provide details for $(\mathcal{P}^j)_j$.

For any given $(\theta, \lambda, \alpha, \tau, p)$, the maximum of the continuous function $V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau); p, \cdot)$ over the non empty compact set \mathbb{Y}_m^j exists. This guarantees the non emptiness of the values of the correspondence \mathcal{P}^j . The compactness of the values of \mathcal{P}^j follows from the continuity of $V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau); p, \cdot)$ and the compactness of \mathbb{Y}_m^j . The convexity of the values of \mathcal{P}^j follows from Assumption (B.1) and the definition of \mathcal{P}^j . Finally, showing that the graph of \mathcal{P}^j is a closed subset is sufficient for the upper hemi-continuity of \mathcal{P}^j since the correspondence takes its values in a compact subset. Let $(z_n)_n = (p_n, q_n, \alpha_n, \lambda_n, x_n, \theta_n, y_n, \tau_n)_n \in \prod_n \mathbb{Z}$ be a sequence of elements that converges to $z = (p, q, \alpha, \lambda, x, \theta, y, \tau)$ and $y_n^j \in \mathcal{P}^j(z_n)$ for all n . Recall that \mathbb{Z} results from the cartesian product of a family of compacts, hence closed subsets. So, each limit point belongs to each relative set. Using the continuity of $\underline{\beta}^j$ and V^j on their domains, then one obtains by passing to the limit that: for all $y^j \in \mathbb{Y}_m^j$, $V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau) p, y^j) \geq V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau) p, y^j)$ which means that $y^j \in \mathcal{P}^j(z)$. Hence, the graph of \mathcal{P}^j is closed. \square

By assumption (A.6), Lemma 4 and the continuity of the function \mathcal{T}^j for each $j \in \mathcal{J}$, the correspondence \mathcal{G} is non-empty and convex valued as well as upper hemi-continuous. By Kakutani's Fixed-Point Theorem, there exists a fixed point $\tilde{z}_m := (\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{\tau}) \in \mathbb{Z}$.

Step 2: A fixed point $\tilde{z}_m := (\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{\tau})$ is a modified pseudo-equilibrium for the truncated economy \mathcal{E}_m .

Lemma 5 For all $s = 0, \dots, S$,

$$\sum_{h \in \mathcal{H}} (\tilde{x}_s^h - w_s^h) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} \tilde{y}_s^j \leq 0,$$

and, for all j ,

$$\sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} - 1 \leq 0.$$

Proof: Summing the consumers' budgets we have that, at a fixed point,

$$\tilde{p}_0 \left(\sum_h (\tilde{x}_0^h - w_0^h - \sum_j \tilde{\theta}^{hj} y_0^j) \right) + \sum_j \tilde{q}^j \left(\sum_h \tilde{\theta}^{hj} - 1 \right) \leq 0.$$

so that the value of the auctioneer's objective in $t = 0$ is non-positive. Similarly for all $s \neq 0$:

$$\tilde{p}_s \left(\sum_h (\tilde{x}_s^h - w_s^h - \sum_j \tilde{\theta}^{hj} y_s^j) \right) \leq 0.$$

The properties stated in the Lemma follow because, if any of them failed, there would exist $((p_0, q), p) \in \Delta_{L+J-1} \times \prod_s \Delta_{L-1}$ giving a positive value to the auctioneer's objectives, contradicting the definition of \mathcal{M} . \square

Thanks to Lemma 5, we can assume that m is big enough so that the consumption of each consumer at the pseudo-equilibrium is in the interior of the hyper-cube $M = [-m, m]^{(L(S+1)H+L(S+1)J)}$. We use this property from now on.

Lemma 6 For all $h \in \mathcal{H}$,

$(\tilde{x}^h, \tilde{\theta}^h)$ solves $\text{Max}_{(x^h, \theta^h)} \{u^h(x^h) \text{ s.t. } (x^h, \theta^h) \in \mathbb{B}^h(\tilde{p}, \tilde{q}, \tilde{y}) \cap [\mathbb{X}_m^h \times \Theta_m^h]\}$, and the components of $\tilde{\lambda}^h \in \mathbb{R}_{++}^{(S+1)}$ are the Lagrange multipliers of the above maximization problem.

Proof: The proof follows a standard argument once we show that, at the fixed point, the budget set has an interior point. By the monotonicity of u^h (A.4), when m is big enough so that \tilde{x}^h is in the interior of the compact set \mathbb{X}_m^h , we must have $\tilde{\lambda}^h \gg 0$. Using this, and the assumption of strictly positive endowments (A.5), we see that the budget defined by $\tilde{\lambda}^h G^h(\tilde{p}, \tilde{q}, \tilde{y}, \tilde{x}^h, \tilde{\theta}^h) \leq 0$ has an interior point. \square

Lemma 7 For all $j \in \mathcal{J}$,

\tilde{y}^j solves $\text{Max}_{y^j} \left\{ V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}); \tilde{p}, y^j) \text{ s.t. } y^j \in \mathbb{Y}_m^j \right\}$ where for all $y^j \in \mathbb{Y}_m^j$, $V^j(\beta^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}); \tilde{p}, y^j) = \sum_s \beta_s^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau})(\tilde{p}_s y_s^j)$, and for all s

$$\beta_s^j(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) := \sum_{h \in \mathcal{H}^j} \beta_s^{hj}(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau})$$

$$\text{with } \beta_s^{hj}(\tilde{\theta}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\tau}) := \frac{\tilde{\lambda}_s^h}{\tilde{\lambda}_0^h} \tilde{\tau}^j \tilde{\theta}^{hj} + \left(\delta^{hj} - \tilde{\tau}^j \tilde{\theta}^{hj} \right) \frac{\tilde{\alpha}_s^{hj}}{\tilde{\alpha}_0^{hj}}.$$

Proof: The proof follows from the definition of \mathcal{P}^j , the observation that $\beta_s^j = (\Pi_h \lambda_0^h \alpha_0^{hj}) \beta_s^j$ for all $s = 0, \dots, S$, and the fact that $\Pi_h \tilde{\lambda}_0^h \tilde{\alpha}_0^{hj} > 0$ by Lemma 6. \square

Lemma 8 *At $s = 0$,*

$$\sum_{h \in \mathcal{H}} (\tilde{x}_0^h - w_0^h) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} \tilde{y}_0^j = 0$$

and for all $j \in \mathcal{J}$, $\tilde{q}^j \left(\sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} - 1 \right) = 0.$ (19)

For all $s \neq 0$,

$$\sum_{h \in \mathcal{H}} (\tilde{x}_s^h - w_s^h) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} \tilde{y}_s^j = 0.$$

Proof: Aggregating over all consumers the binding budget constraints (given Assumption (A.4) and Lemma 6) and using the non-negativity of the prices \tilde{p} and \tilde{q} , one sees that,

$$\tilde{q}^j \left(\sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} - 1 \right) = 0, \quad j \in \mathcal{J}$$

and, for all $s = 0, \dots, S$,

$$\tilde{p}_s \left(\sum_{h \in \mathcal{H}} (\tilde{x}_s^h - w_s^h) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} \tilde{y}_s^j \right) = 0,$$

By free-disposal (B.3), we obtain equalities in the market clearing conditions for commodities. □

This ends the proof of Proposition 3.

PART II.

Proposition 9 *Assume (A.1)–(A.5), (A.6'), (A.7) and (B.1)–(B.4). Then there exists an equilibrium for \mathcal{E}_m .*

Proof of Proposition 9:

If at a modified pseudo-equilibrium $\tilde{q}^j > 0$ for all j , we have market clearing of the asset market, $\sum_h \tilde{\theta}^{jh} = 1$, $\tilde{\tau}^j = 1$ for all j , and a modified pseudo-equilibrium is an equilibrium.

We now argue that if at a modified pseudo-equilibrium for some firm j $\tilde{q}^j = 0$ and $\sum_h \tilde{\theta}^{jh} < 1$, we can always re-scale the production plan, the shareholdings and the price and obtain a full equilibrium.

Consider a modified pseudo equilibrium $(\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{\tau})$ with $\tilde{q}^j = 0$ for some firm j .

First notice that:

$$\sum_s \tilde{\beta}_s^j (\tilde{p}_s \tilde{y}_s^j) = 0. \quad (20)$$

Indeed, under (A.6') and (A.7), the Consistency Lemma implies that $\sum_s \tilde{\beta}_s^j (\tilde{p}_s \tilde{y}_s^j) \leq 0$. We can conclude that the inequality is in fact an equality because otherwise the firm would have done better by stopping activity, which is always a feasible choice.

We now re-scale production plans, portfolios and prices as follows: $\hat{\theta}^{jh} = \tilde{\tau}^j \tilde{\theta}^{jh}$, $\hat{y}_s^j = \frac{1}{\tilde{\tau}^j} \tilde{y}_s^j$, $\hat{q}^j = \frac{1}{\tilde{\tau}^j} \tilde{q}^j$.

Clearly, $\sum_h \hat{\theta}^{jh} = 1$, and we may set $\hat{\tau}^j = 1$. Then, for all j and all s , $\beta_s^j(\hat{\theta}, \tilde{\lambda}, \tilde{\alpha}, \hat{\tau}) = \beta_s^j(\hat{\theta}, \tilde{\lambda}, \tilde{\alpha}, \hat{\tau}) = \hat{\beta}_s^j$ (see eq. (11)).

The re-scaling does not affect the consumer budget, nor the consumers' first order conditions.

Finally, the rescaled production plan \hat{y}^j is feasible ($\hat{\tau}^j \geq 1$, $0 \in Y^j$ and convexity of Y^j) and optimal for firm j at the (unchanged) state prices $\hat{\beta}^j = \tilde{\beta}^j$ ($\hat{\beta}^j \hat{y}^j = \frac{1}{\tilde{\tau}^j} \tilde{\beta}^j \tilde{y}^j = 0$). In other words, the re-scaled plan belongs to the efficient boundary of the production set Y^j at the (unchanged) state prices used by the firm.

We have shown that for the truncated economy \mathcal{E}_m , a modified quasi-equilibrium $\tilde{z}_m = (\tilde{p}, \tilde{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{\tau})$ is transformed into an equilibrium $\hat{z}_m = (\tilde{p}, \hat{q}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \hat{\theta}, \hat{y})$ after re-scaling. \square

PART III.

Proposition 10 *Assume (A.1) – (A.5), (A.6'), (A.7) and (B.1) – (B.4). Then there exists an equilibrium for \mathcal{E} .*

Proof of Proposition 10:

Assumption (B.4) ensures that for m big enough at the equilibrium of the truncated economy, \hat{z}_m , all endogenous variables defining other than ownership shares are in the interior of the hyper-cube M ; moreover, for all m $\sum_h \hat{\theta}_m^{hj} = 1$ for all j and all s , so that the the upper bound on ownership shares is also not binding.

Let m go to infinity. All variables remain in compact sets, so there exists a converging subsequence. Let z^* be its limit. Conditions (1), (2) and (3) in the definition of equilibrium are satisfied at the limit.

By Assumptions (A.2) and (A.4), (x^{h*}, θ^{h*}) is a solution of consumer h unconstrained decision problem, so that condition (4) is also verified. Finally, using Assumption (B.1) and the convexity of the objective function V^j of firm j on \mathbb{Y}^j , y^{j*} is a solution of firm j unconstrained decision problem, (5). \square

Remark: The argument in Part III does not apply to a sequence of modified pseudo-equilibria, because it may be that, for some j , $\sum_h \theta^{hj} = 0$ in the limit, leaving τ^j undefined. The notion of modified pseudo-equilibrium was introduced because the argument in Part II does not apply to a pseudo-equilibrium.

A4. Proof of Theorem 1

The proof follows from the proof of Theorem 3 if one notices that a pseudo equilibrium is a modified pseudo equilibrium in which for all j τ^j is fixed to be identically equal to one, and condition (6) in the definition of pseudo-equilibrium is omitted. The argument in Part I of the proof shows that a pseudo equilibrium for the truncated economy exists, and the argument in Part III can be applied directly to the sequence of pseudo-equilibria.

A5. Proof of Theorem 2

The proof follows the steps of Parts I and III of the proof of Theorem 3, but some important modifications are needed.

Consider the truncated economy \mathcal{E}_m .

We introduce a new set of variables $\pi \in [m, m]^J$. The auctioneer at $t = 0$ now chooses (p_0, π) in the non empty compact convex set

$$\mathbb{P}_0 = \{(p_0, \pi) \in \mathbb{R}_+^L \times \mathbb{R}^J \mid \sum_l p_{0l} = 1, \pi \in [-m, m]^J\}.$$

We define two continuous functions component by component as follows

$$\mathcal{Q}^j(\pi) = \max[0, \pi^j]$$

$$\mathcal{K}^j(\pi) = -(\pi^j + m)$$

$$\text{Def } ((p_0, \pi), (p_s)_{s \geq 1}, q) \in \mathbb{Q}' = \mathbb{P}_0 \times \prod_{s \geq 1} \Delta_{L-1} \times \prod_j [0, m]^j$$

Let $\mathbb{Z}' = \mathbb{Q}' \times \prod_{\mathcal{H}\mathcal{J}} \Delta_S \times \prod_{\mathcal{H}} \Delta_S \times \prod_{h \in \mathcal{H}} \mathbb{X}_m^h \times \prod_{h \in \mathcal{H}} \Theta_m^h \times \prod_{j \in \mathcal{J}} \mathbb{Y}_m^{j'} \times [-2\bar{m}, 0]^J \times \prod_{\mathcal{J}} \mathbb{T}$ and $z = (p, q, \pi, \alpha, \lambda, x, \theta, y, k, \tau)$. The constrained-equilibria correspondence for the truncated economy \mathcal{E}_m is defined as

$$\begin{aligned} \mathcal{G}' : \mathbb{Z}' &\rightrightarrows \mathbb{Z}' & (21) \\ z &\mapsto ((\mathcal{C}^h(z))_h, (\mathcal{P}^{j'}(z))_j, (A^{hj}(z))_{hj}, (\Lambda^h(z))_h, \mathcal{M}'(z), (\mathcal{K}^j(z))_j, (\mathcal{Q}^j(z))_j, (\mathcal{T}^j(z))_j) \end{aligned}$$

where $A^{hj} : \mathbb{Q}' \times \prod_{j \in \mathcal{J}} \mathbb{Y}_m^j \rightarrow \Delta_S$ is the upper hemi-continuous correspondence describing h 's price theory, the correspondences $(\mathcal{C}^h)_h$, $(\Lambda^h)_h$ and $(\mathcal{T}^j(z))_j$ are as in the proof of Theorem 3, the functions $(\mathcal{K}^j(z))_j$ and $(\mathcal{Q}^j(z))_j$ are as defined above, while the firm's and auctioneer correspondences $\{(\mathcal{P}^{j'})_j, \mathcal{M}'\}$ are defined as follows.

For any $j \in \mathcal{J}$, let V^j be the the map defined in the prof of Theorem 3: $V^j(\underline{\beta}^j; p, y^j) = \sum_{s \in \mathcal{S}} \underline{\beta}_s^j(\theta, \lambda, \alpha, \tau)(p_s y_s^j)$. Then, for any $j \in \mathcal{J}$, let

$$\begin{aligned} \mathcal{P}^{j'} : \mathbb{Z}' &\rightrightarrows \mathbb{Y}_m^j & (22) \\ z &\mapsto \text{Argmax} \left\{ \min_l [p_{0l}] V^j(\underline{\beta}^j(\theta, \lambda, \alpha, \tau); p, y^j) \mid y^j \in \mathbb{Y}_m^j(p, k^j) \right\}, \end{aligned}$$

with

$$\mathbb{Y}^j(p, k^j) = \{y^j \in Y^j \mid p_0 y_0^j \geq k^j\}.$$

The role of the term $\min_l [p_{0l}]$ will be clarified later.

The auctioneer's choice correspondence is

$$\begin{aligned} \mathcal{M}' : \mathbb{Z}' &\rightrightarrows \mathbb{Q}' \\ z &\mapsto \mathcal{M}(z) \end{aligned} \tag{23}$$

where

$$\mathcal{M}_0(z) = \mathit{Argmax}_{(p_0, \pi) \in \mathbb{Q}_0} \left\{ p_0 \left(\sum_h (x_0^h - w_0^h - \sum_j \theta^{hj} y_0^j) + \sum_j \pi^j \left(\sum_h \theta^{hj} - 1 \right) \right) \right\}$$

and, for $s \neq 0$,

$$\mathcal{M}_s(z) = \mathit{Argmax}_{p_s \in \Delta_{L-1}} \left\{ p_s \left(\sum_h (x_s^h - w_s^h - \sum_j \theta^{hj} y_s^j) \right) \right\}$$

Proposition 11 *Assume (A.0)–(A.6) and (B.0)–(B.4). Then there exists a fund-constrained equilibrium for \mathcal{E}_m .*

To apply Kakutani's Theorem to the correspondence \mathbb{G}' , we need to show that the auctioneer's correspondences \mathcal{M}' and the supply correspondences $(\mathcal{P}^{j'})_j$ are non empty and convex valued, and upper hemi-continuous. For \mathcal{M}' this follows from standard arguments. We prove the upper hemi-continuity of $(\mathcal{P}^{j'})$. This is where the term $\min_l [p_{0l}]$ in the definition of $\mathcal{P}_j^{j'}$ plays a role¹⁷. At a point z where $p_0 = 0$, $\mathcal{P}_j^{j'} = \mathbb{Y}_m^j$; the upper hemi-continuity of $\mathcal{P}_j^{j'}$ then follows from the next Lemma.

Lemma 12 *The correspondence $\mathbb{Y}^j : \mathbb{Q}_0 \times [-2\bar{m}, 0] \rightarrow \mathbb{Y}_m^j$ is i) upper hemi-continuous and ii) lower hemi-continuous at all (p_0, k^j) with $p_0 \gg 0$.*

Proof: Given the compactness of \mathbb{Y}_m^j , to prove upper hemi-continuity it is enough to show that the graph of the correspondence is closed. Let $(p_{0n})_n$ be a sequence converging to p_0 , $(k_n^j)_n$ a sequence converging to k^j and $(y_n^j)_n$ a sequence converging to y^j , with $(y_n^j)_n \in \mathbb{Y}^j(p_{0n}, k_n^j)$ for all n . Given that $p_{0n} y_{0n}^j \geq k_n^j$ for all n , we have $p_0 y_0^j \geq k^j$, and $y^j \in \mathbb{Y}^j(p_0, k^j)$.

¹⁷We are following an idea of Greenberg (1977).

To prove lower hemi-continuity at point (p_0, k^j) with $p_0 \gg 0$, let $(p_{0n})_n$ be a sequence converging to $p_0 \gg 0$, $(k_n^j)_n$ a sequence converging to k^j and (y^j) an element of $\mathbb{Y}^j(p_0, k^j)$. We have to construct a sequence $(y_n^j)_n$ converging to y^j such that $(y_n^j)_n \in \mathbb{Y}^j(p_{0n}, k_n^j)$ for all n .

We distinguish three cases:

i) $p_0 y_0^j > k^j$.

In this case for n large enough $p_{0n} y_0^j > k_n^j$, and we can use the constant sequence $y_n^j = y^j$.

ii) $p_0 y_0^j = k^j < 0$.

Using the fact that, by (B.1) and (B.3), \mathbb{Y}_m^j is convex and $0 \in \mathbb{Y}_m^j$, and noticing that $p_0 0 > k^j$, we can construct a sequence satisfying the definition of lower hemi-continuity by taking convex combinations between y^j and 0.

iii) $p_0 y_0^j = k^j = 0$. By (B.0), and $p_0 \gg 0$, it must be that $y_0^j = 0$. We can then use the constant sequence $y_n^j = y^j = 0$.

□

The correspondence \mathcal{G}' is non-empty and convex valued as well as upper hemi-continuous. By Kakutani's Theorem, there exists a fixed point.

We now show that, starting from a fixed point $\tilde{z}_m := (\tilde{p}, \tilde{q}, \tilde{\pi}, \tilde{\alpha}, \tilde{\lambda}, \tilde{x}, \tilde{\theta}, \tilde{y}, \tilde{k}, \tilde{\tau})$ we can construct a fund-constrained equilibrium for the truncated economy \mathcal{E}_m .

Lemma 13 *For all $s = 0, \dots, S$,*

$$\sum_{h \in \mathcal{H}} (\tilde{x}_s^h - w_s^h) - \sum_{j \in \mathcal{J}} \sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} \tilde{y}_s^j \leq 0,$$

and, for all j ,

$$\sum_{h \in \mathcal{H}} \tilde{\theta}^{hj} - 1 = 0.$$

Proof: Summing the consumers' budgets at $t = 0$, at a fixed point,

$$\tilde{p}_0 \left(\sum_h (\tilde{x}_0^h - w_0^h - \sum_j \tilde{\theta}^{hj} y_0^j) \right) + \sum_j Q^j(\tilde{\pi}) \left(\sum_h \tilde{\theta}^{hj} - 1 \right) \leq 0.$$

This implies that the value of the auctioneer's objective in $t = 0$ is non-negative. Indeed, using the definition of \mathcal{M}'_0 , we see that for every j , three possibilities arises:

- either $(\sum_h \tilde{\theta}^{hj} - 1) > 0$ and $\tilde{\pi}^j > 0$, so that $Q^j(\tilde{\pi}) = \tilde{\pi}^j$,
- or $(\sum_h \tilde{\theta}^{hj} - 1) = 0$, in which case $Q^j(\tilde{\pi})(\sum_h \tilde{\theta}^{hj} - 1) = \tilde{\pi}^j(\sum_h \tilde{\theta}^{hj} - 1)$.
- or $(\sum_h \tilde{\theta}^{hj} - 1) < 0$ and $\tilde{\pi}^j = -\bar{m}$, in which case $Q^j(\tilde{\pi}) = 0$ and $K^j(\tilde{\pi}) = 0$ so that, using (B.0), $\tilde{p}_0 \tilde{y}_0^j = 0$. But then we could modify \tilde{z}_m so that $(\tilde{x}^h, \theta^{hj'}) \in \mathcal{C}^h(\tilde{z}_m)$ satisfies $(\sum_h \tilde{\theta}^{hj'} - 1) = 0$. Notice that, after the adjustment, $\tau^{j'} = 1$ and the criterion of the firm is unchanged.

Without loss of generality, at a fixed point, for all j , $Q^j(\tilde{\pi})(\sum_h \tilde{\theta}^{hj} - 1) = \tilde{\pi}^j(\sum_h \tilde{\theta}^{hj} - 1)$, and the value of the auctioneer's objective at $t = 0$ is non-positive. Aggregating the time 1 budgets we also have that the value of the auctioneer's objective is non-positive also for all $s \neq 0$. The properties stated in the Lemma follow because, if any of them failed, there would exist $((p_0, \pi), p) \in \mathbb{P}_0 \times \prod_s \Delta_{L-1}$ giving a positive value to the auctioneer's objectives, contradicting the definition of \mathcal{M}' . \square

Thanks to Lemma 13, we can choose a $\bar{m} > 0$ big enough so that for any $m \geq \bar{m}$ the choices of each consumer at the fixed point are in the interior of the the cube $M = [-m, m]^{(L(S+1)H + L(S+1)J)}$. Using this property, we can continue the proof of Proposition 11 as in Part I of Theorem 3. A new aspect is that we need to show that, at a fixed point $p_0 \gg 0$, so that firms are indeed maximizing the objective V^j . But this follows from (A.0) and the fact that for all h , $p_0 w_0^h + \sum_j q^j \delta^{hj} > 0$. Indeed, if for some l , $p_{0l} = 0$, then, by (A.0), $x_{0l}^h = m$, contradicting Lemma 13.

Also, for $m \geq \bar{m}$, if at the fixed point $q^j > 0$, then $k^j < -m \leq -\bar{m} < -p_0 y_0^j$, so that condition (6) in the definition of investment-constrained equilibrium is verified.

To complete the proof of Theorem 2, we let m go to infinity and consider a sequence \tilde{z}_m of investment-constrained equilibria for \mathcal{E}_m . The variables π

are bounded from above. Indeed, if for some j , $\pi_{j,m} \rightarrow \infty$, we would have $q_j \rightarrow \infty$, giving infinite wealth to consumers with $\delta^{hj} > 0$. Under (A.0), this contradicts Lemma 13. On the other hand, it may be that for some j , $\pi_{j,m} \rightarrow -\infty$ or $k_{j,m} \rightarrow -\infty$. In these cases we define $\pi_j = -\bar{m}$ and $k_j = -2\bar{m}$. All other variables except portfolio holdings are in compact sets. Along the sequences $\sum_h \theta^{hj} = 1$ for all j , so that even portfolios remain in a compact set. Thus there exists a limit point \tilde{z} . To show that the limit point is an investment-constrained equilibrium of \mathcal{E} , we have to show that, at the limit, $\tilde{p}_0 \gg 0$, but this follows from (A.0), using the same argument as above. \square