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Asymmetric CAPM Dependence for Large Dimensions: the Canonical Vine Autoregressive Model

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Asymmetric CAPM dependence for large dimensions: the Canonical Vine Autoregressive Model

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Abstract

We propose a new dynamic model for volatility and dependence in high dimensions, that allows for departures from the normal distribution, both in the marginals and in the dependence. The dependence is modeled with a dynamic canonical vine copula, which can be decomposed into a cascade of bivariate conditional copulas. Due to this decomposition, the model does not suffer from the curse of dimensionality. The canonical vine autoregressive (CAVA) captures asymmetries in the dependence structure. The model is applied to 95 S&P500 stocks. For the marginal distributions, we use non-Gaussian GARCH models, that are designed to capture skewness and kurtosis. By conditioning on the market index and on sector indexes, the dependence structure is much simplified and the model can be considered as a non-linear version of the CAPM or of a market model with sector effects. The model is shown to deliver good forecasts of Value-at-Risk.

Keywords: asymmetric dependence, high dimension, multivariate copula, multivariate GARCH, Value-at-Risk.

JEL Classification: C32, C53, G10

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1 Introduction

Taking account of conditional volatilities and dependence is key for solving many financial problems, including optimal portfolios and risk management. A large literature has been devoted to univariate volatility modelling, following Engle (1982). In recent years, attention has shifted to the joint study of large cross-sections of asset returns. The first multivariate models that were proposed in the literature were the Constant Conditional Correlation (CCC) model of Bollerslev (1990), the VEC model of Bollerslev, Engle & Wooldridge (1988) and the BEKK model of Engle & Kroner (1995). For a recent survey of multivariate GARCH models, see Bauwens, Laurent & Rombouts (2006) and Silvennoinen & Tersvirta (2007). Recently, Engle (2002) introduced the Dynamic Conditional Correlation (DCC) model, which has become the benchmark for multivariate volatility models. The approach consists first in using a set of univariate GARCH models for each one of the n assets and then to model the $n(n-1)/2$ pairs of correlations jointly. These approaches work for moderately sized problems (up to 30 stocks in Cappiello, Engle & Sheppard (2006), who propose and estimate asymmetric DCC models for thirty stock and bond indexes). Another strand of the literature uses copulas in order to bring more flexibility to the modeling of dependence. For instance Patton (2006*a*) introduces bivariate models of time-varying copulas that allow for departures from normality and symmetry in the dependence, while Jondeau & Rockinger (2006) work with symmetric time-varying copulas for four stock indexes. Lee & Long (2009) use copulas to model non-linear dependence in the innovations of a DCC model for trivariate data, but they mention how to their approach could be extended to higher dimension. One remaining challenge in multivariate GARCH models is to come up with flexible distributions and models that are also feasible for the joint returns of large sets of assets. Andersen, Bollerslev, Christoffersen & Diebold (2006) study volatility and correlation models in the context of risk management and note that "the viability of [copula] methods in very high-dimensional systems remains to be established" (p. 533).

In this paper we introduce the canonical vine autoregressive (CAVA) model, a dynamic model of dependence based on canonical vine copulas, which we demonstrate to be suitable for large cross-sections of assets (more than 30 assets) with time-varying volatilities and possibly non-Gaussian marginals. The CAVA model is not subject to size limitation, because it does not involve simultaneous estimation over the whole cross-section of assets. Our contribution is threefold. First, to the best of our knowledge, the CAVA is the only model that can be used for high dimensions, while allowing for departures from normality. Second, the model does not suffer from the curse of dimensionality, nor from an incidental parameter problem, inherent to the method of covariance targeting, used for multivariate GARCH models, see Engle, Shephard & Sheppard (2008). Moreover, there is no need to reestimate the model when more assets are taken into consideration. Third, the model can take advantage of an existing factor structure in the cross-section of stock returns in order to capture dependence in a parsimonious way. We now turn to a more detailed discussion of each of these points.

A recent literature tackles the problem of multivariate GARCH models for very large cross-sections of assets. Palandri (2009) estimates dynamic conditional correlations sequentially for a cross-section of 69 NASDAQ stocks. Engle & Kelly (2008), with the Dynamic

Equicorrelation (DECO) model, estimate 500 dynamic correlations under the restriction that they are equal across assets or groups of assets. Engle (2009) proposes the MacGyver estimation method which pools the autoregressive DCC parameter estimates from bivariate DCC models by taking their median value. Engle et al. (2008) propose an alternative estimation method for the DCC, based on selections of pairwise models, which is fast and leads to consistent estimates of the DCC parameters. While it is well-known that financial returns exhibit significant departures from normality[1] like leptokurtosis, skewness and asymmetric dependence,[2] all these models are based on conditionally Gaussian distributions. A separate literature tries to extend multivariate volatility models to non-Gaussian situations. Pesaran & Pesaran (2007) use a multivariate Student t distribution for returns, that was introduced earlier by Harvey, Ruiz & Shephard (1992). Bauwens & Laurent (2005) propose an asymmetric Student t distribution and apply it to the innovations of a DCC model for three exchange rates. Mencía & Sentana (2003) use the generalized hyperbolic distribution for five NASDAQ sectorial indexes. These distributions introduce additional parameters and cannot readily be used in large dimensions. To the best of our knowledge, the CAVA model is the first one that allows realistic modelling of the empirical features of the joint distribution of returns for large cross-sections of assets. Our modeling strategy allows for a great deal of flexibility in the conditional distribution as well as the dynamics. In particular we are not restricted to Gaussian GARCH models and we can accommodate fat tails and asymmetry in the marginal distributions. Moreover time-varying canonical vine copulas allow for departures from a Gaussian dependence structure, like asymmetric dependence and tail dependence.

Most existing models of volatility and correlation suffer in one way or another from the curse of dimensionality. This means that when these models are applied to a large cross-section of assets, the number of parameters that need to be estimated jointly becomes prohibitively large, rendering estimation impossible. Another problem with multivariate GARCH models is the need to ensure that the variance covariance matrix of returns is always positive semidefinite. The DCC avoids this problem, but still suffers from the fact that an n-dimensional variance covariance matrix needs to be inverted for each observation in the evaluation of the likelihood. Moreover, models like the BEKK and the DCC must be based on targeting in order to be estimable in large dimensions ($n > 30$). In the case of the DCC this means that one needs an estimate of the variance covariance matrix of the standardized residuals of the univariate GARCH models and that the DCC parameters are estimated conditionally on these $n(n-1)/2$ parameters. Engle et al. (2008) show that, due to the incidental parameter problem, this leads to strong downward biases in estimates of the autoregressive parameters of the DCC when the number n of assets under consideration gets very large. The CAVA model does not suffer from these problems. The basic idea underlying our model is that of conditioning. This is the building principle for canonical vine copulas. Following Sklar (1959), we decompose the joint distribution of all returns into the product of the marginals and the canonical vine copula, which captures all

[1]Richardson & Smith (1993) find that stock returns and their idiosyncratic risk, as measured by the residuals from the Capital Asset Pricing Model (CAPM) exhibit significant departures from normality, both in the marginal and in the dependence structure.

[2]For evidence on asymmetry, see Longin & Solnik (1995), Longin & Solnik (2001), Ang & Chen (2002), Ang & Bekaert (2002), Das & Uppal (2004) and Patton (2004), amongst others.

the dependence between the returns. Canonical vine copulas were developed by Bedford & Cooke (2002) and introduced into financial modelling by Aas, Czado, Frigessi & Bakken (2009). Chollete, Heinen & Valdesogo (2009) use canonical vines in a regime-switching framework and show that they are capable of capturing the asymmetric dependence that is present in international financial returns. Canonical vines are multivariate copulas obtained as a product of iteratively conditioned bivariate copulas. Therefore using as building blocks bivariate copulas, one can construct a very flexible multivariate canonical vine copula. The CAVA model consists in making each one of these bivariate copulas potentially time-varying. We propose an autoregressive model of the copula parameter that can be applied to any copula in the same way, which facilitates comparison of the time variation across copulas. This adds a lot of flexibility, as the dynamics is not constrained to be identical across pairs of assets, contrary to the DCC model. Moreover, since the model is formulated in terms of bivariate dependence models, there is no need to estimate all the parameters simultaneously. The decomposition of the joint distribution into the product of the marginals and a product of iteratively conditioned bivariate copulas opens the way for a stepwise estimation method that results in a series of low-dimensional optimizations for bivariate dynamic conditional copulas as in Patton (2006*b*). This means that while the estimation effort doubles if the cross-sectional dimension doubles, the complexity remains the same and the estimation just takes twice as long, since we have to carry out twice as many elementary estimations. Thus it suffers neither from the curse of dimensionality nor from the incidental parameter problem, since the estimation is stepwise and conditional; and it does not require any multivariate non-linear constraint, like positive definiteness, since it is constructed on the basis of a well-defined multivariate distribution.

Finally, the CAVA model is designed to take advantage of an existing factor structure in the cross-section of stock returns in order to capture dependence in a parsimonious way. In its full generality, a canonical vine copula among n variables comprises $n(n-1)/2$ bivariate copulas. The CAVA model concentrates attention on the parts of the distribution which are most relevant from a financial point of view, namely the bivariate dependence models of each stock with the market. In building our model we add to the cross-section of stocks the market return and sectorial indexes. We show that once the stock returns are conditioned on the market and the sector returns, most of the dependence has been captured adequately. Our strategy consists in carefully modeling the dependence between each individual stock and the market return as well as the individual stock return and its sector return, conditioned on the market. Thus, in this version, the CAVA model can be viewed as a non-linear and non-Gaussian extension of the Capital Asset Pricing Model (CAPM) model with sectorial effects.

The paper proceeds as follows. Section 2 presents the Canonical Vine autoregressive (CAVA) model. Section 3 presents the stepwise estimation method. Section 4 deals with the model specification. Section 5 presents the empirical results for 95 weekly $S\&P500$ stock returns as well as the results of an evaluation of VaR. Section 6 concludes.

2 The Canonical Vine Autoregressive Model

In this section, we introduce the Canonical Vine Autoregressive model (CAVA). We first provide a brief account of copula theory and canonical vine copulas, which we use to describe the dependence. Then we show how we introduce dynamics in the model. Finally we explain how we adapt a canonical vine to obtain a very flexible factor model for the cross-section of returns, which can be viewed as a time varying and non-linear extension of the Capital Asset Pricing Model (CAPM).

2.1 Canonical Vine Copula

Traditionally in finance, the question of the dependence between returns has been addressed using Pearson's correlation. This is due in part to the central role of the normal distribution in statistics and of the CAPM in finance. The CAPM assumes multivariate normality of returns and measures dependence with correlation. A limitation of Pearson's correlation is that it is implicitly based on the assumption of normality, or more precisely it is only a natural measure of dependence in the elliptical family of distributions. The most prominent members of this family are the multivariate Gaussian and Student t distributions. Another limitation of Pearson correlation is that it only measures linear dependence and therefore misses non-linear relations between variables.[3]

In empirical finance, there is a vast body of literature suggesting that financial returns are not normally distributed. Thus, financial returns might display more intricate types of dependence than what can be captured by the correlation coefficient. One way of accounting for these more flexible types of dependence is through the use of copulas. Copulas are a very flexible tool to model patterns of dependence between variables separately from their marginal distributions, and may be used to model the observed dependence between financial returns.[4] Since we are interested in modeling financial returns, in the remainder we will use a variety of GARCH models with Gaussian, Student and skewed Student t marginal distributions. These models provide the necessary flexibility to capture the main features of returns distributions like heteroscedasticity, leverage, asymmetry and leptokurtosis.

While there exist many bivariate families of copulas, the choice is much more limited for multivariate copulas. Archimedean copulas can be generalized to the multivariate case, but they imply the strong restrictions that the dependence is the same across all pairs of variables. Only the Gaussian and Student t copulas offer the possibility of having differences in the dependence between pairs of variables. Unfortunately the Gaussian copula cannot account for tail dependence, while the Student t copula restricts upper and lower tail dependence to be equal. Actual returns, on the other hand, tend to exhibit more lower than upper tail dependence, which corresponds to the idea that different stocks are more likely to crash together, than to thrive together.

We now describe the family of copulas that we use in this paper. Bedford & Cooke (2002) introduce canonical vine copulas and Aas et al. (2009) and Berg & Aas (2007), whose presentation we follow here, are the first to use them in a financial application. These very flexible multivariate copulas are obtained by a hierarchical construction. The main idea is that a flexible multivariate copula can be decomposed into a cascade of bivariate copulas.

[3]For instance Embrechts, McNeil & Straumann (2002) demonstrate how Pearson correlation can fail to capture dependence adequately.

[4]Copula theory goes back to the work of Sklar (1959). A more detailed account of copulas can be found in Joe (1997), Nelsen (1999) and in Cherubini, Luciano & Vecchiato (2004) who provide a more finance-oriented presentation. For work on copulas as a modeling tool for returns, see Embrechts, Klüppelberg & Mikosch (1997), and Dias & Embrechts (2004).

It is well known that the joint probability density function of n variables $y_1, \ldots, y_n$ can be decomposed without loss of generality by iterative conditioning, as follows:

$$f(y_1, \ldots, y_n) = f(y_1) \cdot f(y_2|y_1) \cdot f(y_3|y_1, y_2) \ldots f(y_n|y_1, \ldots, y_{n-1}).$$

Each one of the factors in this product can be decomposed further using conditional copulas. For instance the first conditional density can be decomposed as the copula function c_{12} linking y_1 and y_2, multiplied by the density of y_2:

$$f(y_2|y_1) = c_{12}(F_1(y_1), F_2(y_2))f_2(y_2),$$

where $F_i(.)$ denotes the cdf of y_i. In the same way, one (among several) possible decomposition of the second conditional density is:

$$f(y_3|y_1, y_2) = c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1))f(y_3|y_1),$$

where $c_{23|1}$ denotes the conditional copula of y_2 and y_3, given y_1. Further decomposing $f(y_3|y_1)$ leads to:

$$f(y_3|y_1, y_2) = c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1))c_{13}(F_1(y_1), F_3(y_3))f_3(y_3).$$

Finally, combining the last expressions, one obtains the joint density of the first three variables in the system as a function of marginal densities and bivariate conditional copulas:

$$\begin{aligned} f(y_1, y_2, y_3) &= f_1(y_1)f_2(y_2)f_3(y_3) \cdot c_{12}(F_1(y_1), F_2(y_2)) \cdot c_{13}(F_1(y_1), F_3(y_3)) \cdot \\ &\quad c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1)). \end{aligned} \tag{1}$$

The copula density can be written as:

$$c\,(F_1(y_1), F_2(y_2), F_2(y_3)) = c_{12}(F_1(y_1), F_2(y_2))c_{13}(F_1(y_1), F_3(y_3))c_{23|1}(F_{2|1}(y_2|y_1), F_{3|1}(y_3|y_1)).$$

Conditional distribution functions are computed using a formula of Joe (1996):

$$F(y|v) = \frac{\partial C_{y,v_j|v_{-j}}(F(y|v_{-j}), F(v_j|v_{-j})}{\partial F(v_j|v_{-j})},$$

where v_{-j} denotes the vector v excluding the component v_j. This decomposition leads to a canonical vine, in which one variable plays a pivotal role, in our example, y_1. In the first stage of the copula we model the bivariate copulas of y_1 with all other variables in the system. Then we condition on y_1, and consider all bivariate conditional copulas of y_2 with all other variables in the system etc. For an n-dimensional set of variables, this leads to the n-dimensional canonical vine copula density:

$$c\,(F_1(y_1), \ldots, F_n(y_n)) = \prod_{j=1}^{n-1} \prod_{i=1}^{n-j} c_{j,j+i|1,\ldots,j-1}(F(y_j|y_1, \ldots, y_{j-1}), F(y_{j+i}|y_1, \ldots, y_{j-1})),$$

where the conditioning set is empty if $j = 1$.

The advantages of a canonical vine copula are immediately apparent: whereas there are only very few flexible multivariate copulas, there exists a wide menu of bivariate copulas. When specifying a canonical vine copula, we can therefore choose each one of the building blocks involved from a very long list, which allows for a very large number of possible copulas. In a sense, this reverses the traditional problem of not having enough multivariate copulas to having too many to choose from!

2.2 Dynamic Dependence

In the previous section we define the canonical vine copula that we use in this paper. Combined with GARCH models for the marginal distributions of returns, a canonical vine provides a very flexible tool to model dependence. In recent years, attention of financial econometricians has turned to models of time-varying correlations. Longin & Solnik (1995) were amongst the first to document that correlations between asset returns are not constant over time. Tse (2000) develops a test for constant correlations in a multivariate GARCH model. Engle (2002) and Tse & Tsui (2002) introduce models that are designed to capture time-varying correlations. One limitation of these models is that they are restricted to the Gaussian distribution if one models a large number of returns. It is well-known that financial returns are characterized by leptokurtosis, skewness and asymmetric dependence. Some models have been suggested in the literature, that are designed to take these features into account in multivariate GARCH models. Bauwens & Laurent (2005) introduce the asymmetric Student t distribution and evaluate its performance in terms of value-at-risk (VaR) on three exchange rates, while Mencía & Sentana (2003) use the generalized hyperbolic distribution with five sectorial indexes of the NASDAQ. Unfortunately these models are often difficult to estimate and therefore not easily applied to a large cross-section of returns.

In this paper we propose to model dependence with a canonical vine copula and we let all the bivariate conditional copulas that compose the canonical vine be potentially time varying. This requires adding time variation to a number of different bivariate conditional copulas. Several approaches have been put forth in the literature for modeling dependence with time varying bivariate copulas. This line of research was started by Patton (2004), Patton (2006*a*), who proposes autoregressive models of the copula parameter. A non-linear function is used, that maps the real line into the domain of the parameter of the copula. In the real line the conditional copula parameter is modeled as an ARMA(1,1) of an innovation term. The fundamental question with all the dynamic copula models is the same: what should the innovation term on the right hand side of the conditional dependence parameter be? The inputs into the copula are a pair of uniform variables $(u_{1,t}, u_{2,t})$, that can be mapped into the real line with the inverse CDF of the normal distribution: $\epsilon_t = (\epsilon_{1,t}, \epsilon_{2,t})$, where $\epsilon_{i,t} = \Phi^{-1}(u_{i,t})$. Since the original variables are uniform $[0, 1]$, $\epsilon_{i,t}$ will be univariate standard normals. Patton (2006*a*) uses the cross-product of past innovations computed over ten periods for the Gaussian and Student t copulas, $\frac{1}{10}\sum_{i=1}^{10} \epsilon_{1,t-i}\epsilon_{2,t-i}$, while he uses the absolute value of the difference between the uniforms for other copulas: $\frac{1}{10}\sum_{i=1}^{10} |u_{1,t-i} - u_{2,t-i}|$. Such a model has the disadvantage that the current value is a non-linear function of lagged values of conditional dependence. Moreover, the dynamics is not easily comparable across copulas. Of course in the Gaussian world, there are some well-known suggestions in the literature: the DCC of Engle (2002), the model of Tse & Tsui (2002), used by Jondeau & Rockinger (2003) for the Student t copula, and the SCC model of Palandri (2009), who applies a linear ARMA to the Fisher transforms of conditional correlation and of an exponentially smoothed version of the cross-product of past innovations.

We choose, instead, to specify the time variation in the same way for all bivariate copulas, and base it on the DCC equations, as follows:

$$Q_t = \Omega(1 - \alpha - \beta) + \alpha\epsilon_{t-1}\epsilon_{t-1}^{'} + \beta Q_{t-1}, \tag{2}$$

where Ω is a symmetric 2×2 matrix, with ones on the diagonal, that can be interpreted as the unconditional variance covariance matrix of ϵ_t, and α and β are the autoregressive parameters satisfying the restrictions $\alpha > 0$, $\beta > 0$, and $\alpha + \beta \leq 1$. We standardize Q_t as in the DCC:

$$R_t = \{diag(Q_t)\}^{-1/2} Q_t \{diag(Q_t)\}^{-1/2}. \quad (3)$$

Now $\rho_t = R_t[1, 2]$, the off-diagonal element of the matrix R_t, is guaranteed to be in the range $[-1, 1]$. We then transform ρ_t into a Kendall's tau:

$$\tau_t = 2 \arcsin(\rho_t)/\pi. \quad (4)$$

The relation between the Kendall's tau and any copula with parameter θ is given by:[5]

$$\tau = 4 \int_{[0,1]^2} C(u, v, \theta) dC(u, v, \theta) - 1. \quad (5)$$

By inverting this relation, the Kendall's tau can be mapped into the parameter of each one of the copulas. This can be done in closed form for all the copulas we use, except for the Frank copula.[6] So at each period t, we transform τ_t, the Kendall's tau corresponding to ρ_t by Equation (4), into θ_t, the coefficient of each one of the copulas that we estimate. The specification of the dynamics is summarized in Figure 1. Equation (4) is the well-known relation between the copula parameter of the Gaussian and Kendall's tau, see for instance Joe (1997). This implies that for every copula that we estimate, the dynamic Kendall's tau is such that ρ_t can be interpreted as the parameter of the Gaussian copula that would prevail, if the copula were indeed Gaussian. The dynamic Gaussian copula obtains if we plug the parameter ρ_t into the density of the Gaussian copula. The same holds true for the Student t copula, except that the inverse CDF of the Student t distribution with the corresponding degrees of freedom replaces the inverse CDF of the Gaussian in the calculation of ϵ_t. Equations (2), (3) and (4), imply that ρ_t, the parameter of the Gaussian copula follows the dynamic equations of the DCC.[7] Moreover, if the GARCH innovations are Gaussian and we use the Gaussian copula we recover the model of Palandri (2009), but with DCC dynamics.

There is a potential problem with the approach that we describe above. Some copulas are restricted to have positive dependence, and in that case we replace q_t, the off-diagonal element of Q_t by $max(q_t, 0)$. This is a somewhat ad hoc way to guarantee that the model is well-defined in all situations. We do not view this as a problem, since, in general, we do not end up selecting such models. In situations where the dependence is negative during some periods, copulas that also allow for negative dependence fare much better, which translates into higher values of the likelihood. It is important to realize that the non-linear transformations from ρ_t to the copula parameters are part of the way in which we specify

[5] See for instance Embrechts et al. (2002).

[6] In the case of the Gumbel copula or its rotated version the transformation is $\theta_t = 1/(1 - \tau_t)$, in the case of the Clayton, the transformation is $\theta_t = 2/(1 - \tau_t)$. For the Frank, the parameter obtains by solving numerically for the solution of $\tau_t = 1 - \frac{4}{\theta_t}[1 - D_1(\theta_t)]$, where $D_k(x)$ is the Debye function, which is defined for any positive integer k by $D_k(x) = \frac{k}{x^k} \int_0^x \frac{t^k}{\exp(t)-1} dt$.

[7] An alternative possibility is to use the Tse & Tsui (2002) approach.

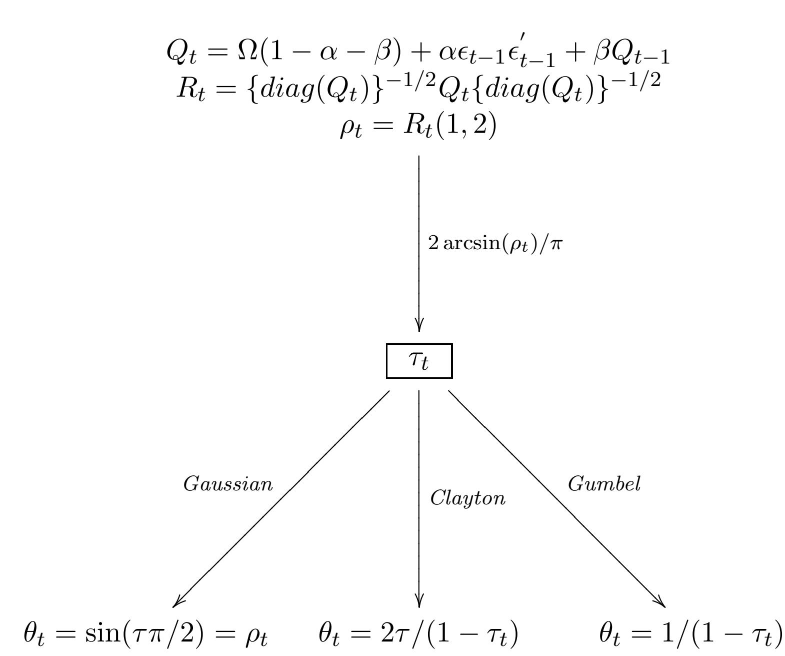

Figure 1: Dynamics of the copula parameter

This figure shows how we specify the dynamics in the copula parameters. First, the inputs into the copula are a pair of uniform variables $(u_{1,t}, u_{2,t})$, that can be mapped into the real line with the inverse CDF of the normal distribution: $\epsilon_t = (\epsilon_{1,t}, \epsilon_{2,t})$, where $\epsilon_{i,t} = \Phi^{-1}(u_{i,t})$. For the case of the Student t copula we use the inverse CDF of the Student t distribution with the corresponding degrees of freedom to calculate ϵ_t. Then we calculate, $Q_t = \Omega(1-\alpha-\beta) + \alpha\epsilon_{t-1}\epsilon_{t-1}' + \beta Q_{t-1}$. We standardize Q_t like in the DCC: $R_t = \{diag(Q_t)\}^{-1/2} Q_t \{diag(Q_t)\}^{-1/2}$. Now, the off-diagonal element of the matrix R_t, ρ_t is transformed into a Kendall's tau according to $\tau_t = 2\arcsin(\rho_t)/\pi$. Finally, we use the fact that there is one to one relationship between Kendall's tau at time t, τ_t, and the parameters of the copula at time t, θ_t, given by $\tau_t = 4\int_{[0,1]^2} C(u,v,\theta_t)dC(u,v,\theta_t) - 1$, to transform the corresponding Kendall's tau, τ_t into the coefficient of each one of the copulas that we estimate, θ_t.

the dynamics. We carry out estimation of the parameters of the model for each copula, but for a given pair of returns, the estimated coefficients Ω, α and β, tend to be quite similar across copulas. This is not surprising and is comparable to what happens to the parameters of a GARCH estimated with different distributional assumptions.

2.3 The Dependence Structure: a Nonlinear CAPM

In the previous section we describe the canonical vine copula that we are using in this paper. In a small dimensional case, one could simply use a full canonical vine and decompose the joint distribution to the fullest level of generality. This is the strategy followed by Aas et al. (2009) in a static non Gaussian case with five series, and by Palandri (2009) in a time varying Gaussian model. This leads to estimating $n(n-1)/2$ bivariate copulas for a cross section of n stocks. In the empirical application of this paper, this means we would have to estimate the parameters of $(95)(94)/2 = 4465$ bivariate copulas! Given the size of the problem at hand we prefer to follow an alternative strategy and devote more modeling effort to those parts of the model that matter most.

Financial theory has devoted a lot of attention to the cross-section of asset returns, and in particular to factor models. The most prominent model in that vein is the CAPM, which implies that the cross section of returns can be explained by a single factor, the market return. This suggests that the most relevant bivariate dependence models in the cross-section of assets are between each asset and the market. The traditional CAPM assumes joint normality of individual returns $r_{i,t}$, of the market return $r_{M,t}$ and of the idiosyncratic cross-sectionally and serially independent error terms $\varepsilon_{i,t}$. This implies the following linear relation:

$$r_{i,t} = \beta_i r_{M,t} + \varepsilon_{i,t}.$$

There has been evidence that the factor loading β_i varies over time, see amongst others Ang & Chen (2007) and more recently Kristensen & Ang (2009). Therefore in the simplest version of our model, we estimate potentially time-varying bivariate copula models for the dependence between each asset and the market, and we consider that conditionally on this, the returns can be characterized by a static Gaussian copula.

Our model goes beyond a certain number of limitations of the CAPM, like non-normality and non-linear dependence, that financial economists have long been aware of. For instance Richardson & Smith (1993) show that asset returns and residuals from CAPM exhibit significant departures from normality. We explicitly allow for departures from the normal distribution, both via more general marginal distributions and in the dependence structure via the use of copulas.

In the context of CAPM, alternative measures of risk, like semivariance, which focuses on the variability of returns on the down side was proposed by Hogan & Warren (1974), and Bawa & Lindenberg (1977) propose a downside beta as a measure of risk. Ang, Chen & Xing (2006) show that stocks with high downside risk, as measured by downside beta, tend to have higher returns in a cross-section. Hong, Tu & Zhou (2007) propose a statistic, based on exceedance correlation to test for asymmetric dependence. Ang & Chen (2002) use a regime-switching model to document the existence of asymmetries in the correlation of individual stocks and portfolios with the market. Correlations conditional on positive returns are typically found to be lower than correlations conditional on negative returns. This non-linear dependence between stocks and the market return is captured in our model with asymmetric lower tail dependent copulas. In addition, whereas CAPM assumes that the idiosyncratic error terms are independent, we model the remaining dependence conditionally on the market with a Gaussian copula. In that sense our model can be viewed as an extension of the CAPM to non Gaussian GARCH marginals with non-linear possibly time-varying dependence captured by a copula.

We propose two versions of the model. The first model, which we call the market model is a non-linear equivalent of a CAPM model, where each stock return has some potentially non-linear, dynamic and asymmetric dependence with the market. In order to clarify the structure of the model, we show the joint density implied by this model in the case of four stock returns, taken from two different sectors, S_1 and S_2. Denote by r_M, the market return, $r_1^{S_1}$ and $r_2^{S_1}$ the returns of assets belonging to sector 1, and $r_1^{S_2}$ and $r_2^{S_2}$ the returns of assets belonging to sector 2. The market model implies the following joint distribution for the four stocks and the market returns:

$$
\begin{aligned}
&f(r_M, r_1^{S_1}, r_2^{S_1}, r_1^{S_2}, r_2^{S_2}) = \\
&\quad f(r_M)\cdot f(r_1^{S_1})\cdot f(r_2^{S_1})\cdot f(r_1^{S_2})\cdot f(r_2^{S_2}) && \textit{Marginals}\\
&\quad \left.\begin{aligned}&\cdot c_{M,r_1^{S_1}}\left(F(r_M),F(r_1^{S_1})\right)\cdot c_{M,r_2^{S_1}}\left(F(r_M),F(r_2^{S_1})\right)\\ &\cdot c_{M,r_1^{S_2}}\left(F(r_M),F(r_1^{S_2})\right)\cdot c_{M,r_2^{S_2}}\left(F(r_M),F(r_2^{S_2})\right)\end{aligned}\right\} && \textit{Market}\\
&\quad \cdot c_{r_1^{S_1},r_2^{S_1},r_1^{S_2},r_2^{S_2}|r_M}\left(F(r_1^{S_1}|r_M),F(r_2^{S_1}|r_M),F(r_1^{S_2}|r_M),F(r_2^{S_2}|r_M)\right). && \textit{Idiosyncratic}
\end{aligned}
\tag{6}
$$

where $c_{M,r_i^{S_j}}$ is the copula between the market and the i-th stock belonging to sector j, and $c_{r_1^{S_1},r_2^{S_1},r_1^{S_2},r_2^{S_2}|r_M}$ is the multivariate Gaussian copula for the dependence between the stocks conditioning on the market.

A generalization of this model consists in enriching the factor structure with sector-specific indexes. Denote by r_{S_1} and r_{S_2} the returns of sector 1 and sector 2 respectively. In this configuration, each stock is assumed to depend on the market and on its own sector return. Again, the dependence considered here is potentially non-linear or asymmetric and is more general than under a standard CAPM. With respect to a full canonical vine structure, we assume that conditionally on the market, the stocks are independent of sector returns other than their own. Moreover we assume that conditionally on the market, the sectorial returns are independent. This second model, which we call the market sector model implies the following decomposition of the joint density of stock, sector and market returns:

$$
\begin{aligned}
&f(r_M, r_{S_1}, r_{S_2}, r_1^{S_1}, r_2^{S_1}, r_1^{S_2}, r_2^{S_2}) = \\
&\quad f(r_M)\cdot f(r_{S_1})\cdot f(r_{S_2})\cdot f(r_1^{S_1})\cdot f(r_2^{S_1})\cdot f(r_1^{S_2})\cdot f(r_2^{S_2}) && \textit{Marginals}\\
&\quad \left.\begin{aligned}&\cdot c_{M,r_{S_1}}\left(F(r_M),F(r_{S_1})\right)\cdot c_{M,r_{S_2}}\left(F(r_M),F(r_{S_2})\right)\\ &\cdot c_{M,r_1^{S_1}}\left(F(r_M),F(r_1^{S_1})\right)\cdot c_{M,r_2^{S_1}}\left(F(r_M),F(r_2^{S_1})\right)\\ &\cdot c_{M,r_1^{S_2}}\left(F(r_M),F(r_1^{S_2})\right)\cdot c_{M,r_2^{S_2}}\left(F(r_M),F(r_2^{S_2})\right)\end{aligned}\right\} && \textit{Market}\\
&\quad \left.\begin{aligned}&\cdot c_{S_1,r_1^{S_1}|r_M}\left(F(r_{S_1}|r_M),F(r_1^{S_1}|r_M)\right)\cdot c_{S_1,r_2^{S_1}|r_M}\left(F(r_{S_1}|r_M),F(r_2^{S_1}|r_M)\right)\\ &\cdot c_{S_2,r_1^{S_2}|r_M}\left(F(r_{S_2}|r_M),F(r_1^{S_2}|r_M)\right)\cdot c_{S_2,r_2^{S_2}|r_M}\left(F(r_{S_1}|r_M),F(r_2^{S_2}|r_M)\right)\end{aligned}\right\} && \textit{Sectors}\\
&\quad \cdot c_{r_1^{S_1},r_2^{S_1},r_1^{S_2},r_2^{S_2}|r_M,r_{S_1},r_{S_2}}(\cdot). && \textit{Idiosyncratic}
\end{aligned}
\tag{7}
$$

where $c_{M,r_{S_j}}$ is the copula between the market and the sector j, $c_{S_j,r_i^{S_j}|r_M}$ is the copula between sector j and stock i of the same sector conditioning on the market and $c_{r_1^{S_1},r_2^{S_1},r_1^{S_2},r_2^{S_2}|r_M,r_{S_1},r_{S_2}}$ is the multivariate Gaussian copula that models the dependence between the stocks conditioning on the market and sectors.

At this point it is important to notice that the market model and the market sector model have some common parts, namely the marginals for the market and the stocks, as well as the bivariate copulas between the market and the stocks. Another possibility, that we leave for further work, is to explore the three classical Fama & French (1992) factors instead of the market sector structure.

3 Stepwise Estimation

In this section we develop the stepwise procedure that we use to estimate the market sector model. We use a 4-step procedure that is a straightforward extension of a two-step procedure that has been studied in a time series copula context by Patton (2006*a*), but that also underlies the estimation of the DCC model as explained in Engle & Sheppard (2001). Both cases are applications of general theorems of Newey & McFadden (1994). We assume that the marginal distribution functions and the joint distribution function are continuous and sufficiently smooth for all required derivatives to exists.

Equation (7) shows that we can decompose the joint density function into four parts, one for the marginals, one for the dependence with the market, one that reflects the dependence structure of the stocks with the own sector conditioning on the market, and finally, one for the multivariate Gaussian copula that we use to model the dependence between the stocks conditioning on the market and the sectors. This decomposition also applies to the log-likelihood function, and this is important, since it opens the way for a stepwise estimation procedure. Without loss of generality we assume that there are J sectors and in each sector there are I stocks. We denote by r_{S_j} the return of sector j and $r_i^{S_j}$ refers to the return of the i-th stock belonging to sector j. Define $R_M = \{r_{M,t}\}_{t=1}^T$, $R_{S_j} = \{r_{S_j,t}\}_{t=1}^T$, $R_i^{S_j} = \{r_{i,t}^{S_j}\}_{t=1}^T$ and $\mathbf{R} = (R_M, R_{S_1}, R_{S_2}, \cdots, R_{S_J}, R_1^{S_1}, \cdots, R_I^{S_J})$. The log-likelihood $L(\mathbf{R}, \Omega)$ is obtained as a sum of the log-likelihoods of the four components:

$$L(\mathbf{R}, \Omega) = L_1(\mathbf{R}; \alpha) + L_2(\mathbf{R}; \alpha, \Theta_M) + L_3(\mathbf{R}; \alpha, \Theta_M, \Theta_S) + L_4(\mathbf{R}; \alpha, \Theta_M, \Theta_S, \Theta_G),$$

$$\begin{aligned}
L_1(\mathbf{R}, \alpha) &= \textstyle\sum_{t=1}^T \left(log(f(r_{M,t}; \alpha_M)) + \sum_{j=1}^J log(f(r_{S_j,t}; \alpha_{S_j})) \right) \\
&\quad + \textstyle\sum_{t=1}^T \sum_{j=1}^J \sum_{i=1}^I log(f(r_{i,t}^{S_j}; \alpha_i^{S_j})), \\
L_2(\mathbf{R}, \alpha, \Theta_M) &= \textstyle\sum_{t=1}^T \sum_{j=1}^J log(c_{M,r_{S_j}}(F(r_{M,t}; \alpha_M), F(r_{S_j,t}; \alpha_{S_j}); \theta_{M,S_j})) \\
&\quad + \textstyle\sum_{t=1}^T \sum_{j=1}^J \sum_{i=1}^I log(c_{M,r_i^{S_j}}(F(r_{M,t}; \alpha_M), F(r_{i,t}^{S_j}; \alpha_i^{S_j}); \theta_{M,i}^{S_j})), \\
L_3(\mathbf{R}, \alpha, \Theta_M, \Theta_S) &= \\
&\textstyle\sum_{t,j,i}^{T,J,I} log(c_{S_j, r_i^{S_j}|r_M}(F(r_{S_j,t}|r_M; \alpha_M, \alpha_{S_j}, \theta_{M,S_j}), F(r_{i,t}^{S_j}|r_M; \alpha_M, \alpha_i^{S_j}, \theta_{M,i}^{S_j}); \theta_{S_j,i})), \\
L_4(\mathbf{R}; \alpha, \Theta_M, \Theta_S, \Theta_G) &= \textstyle\sum_{t=1}^T log(c_{G|r_M, r_{S_1}, \cdots, r_{S_J}}(\cdot; \theta_G)),
\end{aligned}$$

where $\alpha = (\alpha_M, \alpha_{S_1}, \cdots, \alpha_{S_J}, \alpha_1^{S_1}, \cdots, \alpha_I^{S_J})$, are the parameters of the marginals models, $\Theta_M = (\theta_{M,S_1}, \cdots, \theta_{M,S_J}, \theta_{M,1}^{S_1}, \cdots, \theta_{M,I}^{S_J})$ are the parameters of the bivariate copulas for the dependence with the market, while $\Theta_S = (\theta_{S_1,1}, \theta_{S_1,2}, \cdots, \theta_{S_J,I})$ are the parameters of the dependence structure of the stocks with the own sector, conditioning on the market, and Θ_G is the parameter of the multivariate Gaussian copula of the stocks conditional on the market and the sectors. Finally $\Omega = (\alpha, \Theta_M, \Theta_S, \theta_G)$, is the collection of all parameters.

The method proceeds sequentially, as follows:

$$
\begin{aligned}
\hat{\alpha} &= \underset{\alpha}{\text{argmax}}\ L_1(\mathbf{R};\alpha),\\
\hat{\Theta}_M &= \underset{\Theta_0}{\text{argmax}}\ L_2(\mathbf{R};\hat{\alpha},\Theta_M),\\
\hat{\Theta}_S &= \underset{\Theta_S}{\text{argmax}}\ L_3(\mathbf{R};\hat{\alpha},\hat{\Theta}_M,\Theta_S),\\
\hat{\Theta}_G &= \underset{\Theta_G}{\text{argmax}}\ L_4(\mathbf{R};\hat{\alpha},\hat{\Theta}_M,\hat{\Theta}_S,\Theta_G).
\end{aligned}
$$

At each step, estimation is carried out conditionally on the parameters estimated in earlier steps. The central part of the model is the dependence with the market, and those parameters are only conditioned on the marginals, like the parameters of the DCC. The most conditioning that can occur is in the last step, when estimating the Gaussian copula. This contrasts with the method of Palandri (2009), where the conditioning is done with respect to an ever increasing set of parameters, with as many steps as the dimension of the problem. His SCC model is essentially a full decomposition of the canonical vine under normality. Our method is subject to much less snowballing of estimation error, since we condition on at most three previous steps. The market model without sectors obtains by leaving out the third step in the procedure. This estimation is straightforward and can itself be further separated. When we estimate the parameters of the marginals, α, this can be done individually:

$$
\begin{aligned}
\hat{\alpha}_M &= \underset{\alpha_M}{\text{argmax}}\ L_1(\mathbf{R};\alpha_M),\\
\hat{\alpha}_{S_1} &= \underset{\alpha_{S_j}}{\text{argmax}}\ L_1(\mathbf{R};\alpha_{S_j}), \qquad \text{for } j=1,\cdots,J\\
\hat{\alpha}_i^{S_j} &= \underset{\alpha_i^{S_j}}{\text{argmax}}\ L_1(\mathbf{R};\alpha_i^{S_j}), \qquad \text{for } j=1,\cdots,J \text{ and } i=1,\cdots,I.
\end{aligned}
$$

We then collect all the estimates into a single vector $\hat{\alpha}=(\hat{\alpha}_M,\hat{\alpha}_{S_1},\cdots,\hat{\alpha}_{S_J},\cdots,\hat{\alpha}_I^{S_J})$. The same holds when we estimate the dependence structure with the market:

$$
\begin{aligned}
\hat{\theta}_{M,S_j} &= \underset{\theta_{M,S_j}}{\text{argmax}}\ L_2(\mathbf{R};\hat{\alpha},\theta_{M,S_j}), \qquad \text{for } j=1,\cdots,J\\
\hat{\theta}_{M,i}^{S_j} &= \underset{\theta_{M,i}^{S_j}}{\text{argmax}}\ L_2(\mathbf{R};\hat{\alpha},\theta_{M,i}^{S_j}), \qquad \text{for } j=1,\cdots,J \text{ and } i=1,\cdots,I.
\end{aligned}
$$

We collect the estimated parameters into a vector: $\hat{\Theta}_M=(\hat{\theta}_{M,S_1},\hat{\theta}_{M,S_2},\cdots,\hat{\theta}_{M,I}^{S_J})$. Given the previous estimates, we can estimate the parameters of the dependence of the stocks with the own sector, as follows:

$$
\hat{\theta}_{S_{j,i}} = \underset{\theta_{S_{j,i}}}{\text{argmax}}\ L_3(\mathbf{R};\hat{\alpha},\hat{\Theta}_0,\theta_{S_{j,i}}), \qquad \text{for } j=1,\cdots,J \text{ and } i=1,\cdots,I.
$$

This means that the whole model can be decomposed into a series of estimations for the marginals and for iteratively conditioned bivariate copulas.[8] Each one of these bivariate copula estimations is easy to carry out. This brings about important advantages of the model and of its estimation procedure. First, the complexity of the model does not depend on its size, contrarily to what happens with most MGARCH models. Second, the estimation effort is linear in the number of stocks. Finally, the model does not need to be reestimated as more stocks are added.

4 Model Specification

Our framework allows for a great deal of flexibility in the specification of the model. The building blocks of the model are marginal distributions for every asset we consider and bivariate copulas for the dependence of each asset with the market in the case of the market model, and of each asset with its sector-specific return, conditionally on the market in the case of the market sector version of the model. We choose each one of these blocks from a list of possible models. Although the models that are not nested, we use the Bayesian Information Criterion (BIC) which includes a penalty term for the number of parameters and will lead us to choose more parsimonious models.[9] In the remainder of this section we first list the GARCH models that we consider, followed by the bivariate copulas that we use as building blocks for the canonical vine copula. The exact specifications of the GARCH models and bivariate copulas appear in Sections 7.1 and 7.2, respectively.

4.1 GARCH Models for the Marginal Distributions

For the marginal models we follow Cappiello et al. (2006) in selecting the best GARCH model from a list of possible models using Bayesian Information Criterion. We estimate models with Gaussian innovations, but also with the Student-t, as well as the skewed Student t distribution of Hansen (1994). The volatility specifications we consider are: GARCH [Bollerslev (1986)], GJR-GARCH [Glosten, Jagannathan & Runkle (1993)], ZARCH [Zakoian (1994)], AVGARCH [Taylor (1986)], EGARCH [Nelson (1990) Nelson (1991)], APARCH [Ding, Granger & Engle (1993)] and NARCH [Higgins & Bera (1992)].

4.2 Bivariate Conditional Copulas

The conditional bivariate copulas are the building blocks we use in order to construct a large dimensional canonical vine copula. The bivariate copulas that we consider are: Gaussian, Student t, Frank, Gumbel, rotated Gumbel and Clayton. We also consider some mixture copulas: mixture of Gaussians, Gaussian and rotated Gumbel, Gaussian and Frank, Gumbel and rotated Gumbel and Frank and rotated Gumbel.

[8] This further reduces the number of parameters that one is conditioning on at each step. For instance, the copula parameter between the market and a stock is estimated conditionally on only the marginal of the market and of that stock.

[9] Possible alternatives are the Vuong (1989) likelihood ratio test or the Clarke (2007) distribution-free test for non-nested hypotheses. However these tests are designed for pairwise comparison of models, and they can lead to rejecting both models or not ejecting any. This makes them difficult to use in our context, since we are looking to order alternative models and in particular to use a procedure that always delivers an unambiguously preferred model.

ENERGY	14.01	INDUSTRIAL	11.74	HEALTH	11.25	FINANCIAL	17.34	UTILITIES	3.57
XOM	27.84	GE	22.73	JNJ	13.78	BAC	8.14	EXC	12.91
CVX	11.62	UTX	5.06	PFE	9.94	JPM	7.65	SO	6.36
COP	8.01	BA	4.54	MRK	6.05	C	6.29	FPL	6.17
SLB	7.17	MMM	3.80	ABT	5.85	AIG	5.70	D	5.84
OXY	4.24	CAT	3.58	WYE	4.41	WFC	4.81	DUK	5.38
ESV	0.55	PLL	0.30	MYL	0.29	HBAN	0.17	TEG	0.83
BJS	0.49	AW	0.29	MIL	0.28	SOV	0.13	TE	0.82
SUN	0.31	R	0.29	PKI	0.23	MBI	0.11	PNW	0.78
RDC	0.25	CTAS	0.28	WPI	0.22	FHN	0.10	CMS	0.77
TSO	0.20	RHI	0.27	THC	0.21	MTG	0.07	GAS	0.40

MATERIALS	3.61	CONS DISCR	8.60	CONS STAP	10.33	IT	16.14	TELECOM	3.41
DD	10.00	MCD	6.56	PG	16.07	MSFT	11.93	T	56.62
DOW	8.61	CMCSA	6.20	WMT	10.43	IBM	8.53	VZ	26.48
AA	6.93	DIS	5.98	KO	9.13	AAPL	8.23	S	6.21
PX	6.58	TWX	5.49	PEP	8.61	CSCO	7.87	CTL	0.90
NUE	5.10	HD	4.65	CVS	4.66	INTC	6.85	CZN	0.82
SEE	0.90	KBH	0.18	BF.B	0.47	TLAB	0.11		
IFF	0.77	LIZ	0.17	TSN	0.38	NOVL	0.11		
ASH	0.70	JNY	0.14	WFMI	0.37	CPWR	0.11		
BMS	0.60	MDP	0.12	MKC	0.37	QLGC	0.10		
HPC	0.50	DDS	0.11	STZ	0.29	UIS	0.07		

Table 1: Sample Composition and Market Capitalization shares

This table lists the S&P500 stocks that we use in the sample. The sample contains 10 stocks from each of the 10 sectorial indexes that compose the S&P500, except for the Telecom sector where only 5 stocks were present during the whole sample period. For each sector we pick the 5 largest and the 5 smallest stocks in the sector, based on market capitalization shares of June 2008.

For each of these copulas we estimate both a static and a dynamic version, except for the mixture copulas where we only consider the static case, and for each pair of returns we use the BIC to choose the best of all static and dynamic copulas.

5 Empirical Results

5.1 Data

We use a data set of returns of 95 stocks from the S&P500 downloaded from Datastream. The stocks are chosen from 10 different sectors, for which Standard and Poor reports returns. The sector indexes are *Energy, Industrials, Health, Financials, Utilities, Materials, Consumer Discretionary, Consumer Staples, Information Technology and Telecom.* For each sector we pick the 5 largest and the 5 smallest stocks in terms of market capitalization data of June 2008, except for the Telecom sector, where there are only the 5 stocks that are present during all of our sample period. We use weekly data from January 1, 1995 to June 30, 2008, which gives us 703 returns. In addition to the stocks we use the S&P500 and the 10 sectorial stock indexes. Table 1 lists the tickers of the stocks we include as well as their market capitalization. For sectors the market capitalization is given as a fraction of the S&P500, while the individual stocks are quoted as a fraction of their sector.

	GARCH	GJR	ZARCH	AVGARCH	EGARCH	APARCH/NARCH	Total
Gaussian	4	1	0	2	1	0	8
Student t	33	5	12	22	12	0	84
Skew-Student t	3	4	2	2	3	0	14
Total	40	10	14	26	16	0	106

Table 2: Summary results of the GARCH models selected by BIC criterion
This table summarizes the results of the marginal models. It breaks up the results into type of GARCH and distribution selected by the BIC criterion.

5.2 Estimation Results

As in the DCC model, the first stage of the estimation consists in selecting the best marginal model for each of the 106 return series: the $S\&P500$ index return, the 10 sectorial index returns and the 95 stocks. A summary of the result is shown in Table 2, and the detailed result is available in the Supplemental material, in Table 10. The main difference compared to the DCC model is that we are no longer bound to the Gaussian distribution and we can allow for fatter tails with the Student t and additionally for skewness with the skewed Student t. We use a autoregressive moving average model with up to two lags for the conditional mean and select using the BIC criterion. Only 17 out of 106 series exhibit some dynamics, with AR(1) models in all cases but one, where we use an AR(2), and they are mostly sectorial index returns or big stocks. Turning to the GARCH models, we observe that the innovations are distributed as Student-t in 84 cases, where the degrees of freedom go from a minimum of 3.79 for SEE to a maximum of 12.66 for SLB. For the remaining cases the innovations are distributed as skewed Student t for 14 cases and we select a Gaussian innovation in only 8 cases.

The skewness parameters we estimate are negative in all cases except one, which implies that the distributions are negatively skewed, a well-known stylized fact of stock returns. Leaving out $MSFT$, the values of the skewness parameter range from -0.29 for the $S\&P500$ index return to -0.12 for WPI. This asymmetry can be due to aggregation[10] and, not surprisingly, it is found mostly in the indexes: the $S\&P500$ and the *Energy*, *Industrials*, *Health* and *Financials* indexes. These results show the limits of using the Gaussian as marginal distribution for the returns. Clearly, the Student t is preferred, but it does not solve all the problems, since for some of the more aggregate series, taking skewness into account is also important. As far as the dynamics of the conditional variance in concerned, we observe that the standard GARCH model is selected in 40 cases, followed by the AV-GARCH model in 26 cases. In 38% cases the selected GARCH model presents some sort of leverage effect: 16 cases for the EGARCH, 14 for ZARCH and 10 for the GJR-GARCH specification. APARCH and NARCH are never selected, but some models we select are special cases of these more general specification. Given that we use the BIC criterion, that is quite conservative, this result is not surprising, as we tend to select parsimonious models. Another remarkable feature of the results is that there is clustering in the leverage effect: most of the stocks in the *Industrials* and *Financials* sectors present a leverage effect, while

[10] Let X and Y be two random variables with symmetric marginals but with an asymmetric underlying copula. Then the density function of $X + Y$ is asymmetric.

	MARKET	ENE.	IND.	HEAL.	FIN.	UTIL.	MAT.	C. DISCR	C. STAP	IT	TEL.	Total
Independence	0	0	0	2	0	0	0	0	2	0	0	4
Gaussian	13	3	2	3	5	5	2	4	0	2	1	40
Student-t	6	0	1	1	3	2	1	1	3	1	1	20
Frank	22	2	3	1	0	0	3	5	1	4	1	42
Gumbel	0	0	2	2	0	0	0	0	2	0	0	6
Rgumbel	14	0	1	0	0	0	0	0	0	0	0	15
Clayton	1	0	0	0	0	0	0	0	0	0	1	2
Mixture Copulas												
Gaussian-Gaussian	0	0	0	0	0	0	0	0	0	0	0	0
Gaussian-Rgumbel	0	0	0	0	0	0	0	0	0	0	0	0
Gumbel-Rgumbel	0	1	0	1	0	0	0	0	0	0	0	2
Frank-Rgumbel	5	0	0	0	0	0	0	0	0	0	0	5
Rgumbel-Rgumbel	0	0	0	0	0	0	0	0	0	0	0	0
Gaussin-Frank	1	0	0	0	1	1	0	0	0	1	0	4
Time varying Copulas												
Gaussian tv	18	1	1	0	0	0	3	0	2	2	0	27
Student-t tv	5	1	0	0	1	2	0	0	0	0	1	10
Frank tv	9	2	0	0	0	0	1	0	0	0	0	12
Gumbel tv	0	0	0	0	0	0	0	0	0	0	0	0
Rgumbel tv	11	0	0	0	0	0	0	0	0	0	0	11
Clayton tv	0	0	0	0	0	0	0	0	0	0	0	0
% Time Varying	41	40	10	0	10	20	40	0	20	20	20	30
% Asymetry	30	10	30	30	0	0	0	0	20	0	20	20

Table 3: Summary results of the bivariate copulas selected by BIC criterion

only two stocks in the *Utilities* sector have leverage, and almost none of the stocks in the *Health* and *Utilities* sectors present a significant leverage effect.

Having selected the marginal models for all series, we move to selecting all the bivariate copulas that we use as components of the canonical vine copula. Tables 3 and 4 show the results of the dependence structure of the data. The left panel of Table 4 contains the copulas for the dependence of each sector or stock with the $S\&P500$ index return, while the right panel contains the copulas that capture the dependence between every stock and its sectorial return, conditional on the $S\&P500$ index return. The Gaussian copula is not always selected, which is an indication that the Gaussian dependence which is underlying the traditional DCC is not appropriate. The same applies with the Student t copula. A multivariate Student t copula implies that all the dependence models with the market are Student t copulas with the same degrees of freedom for each stock or sectorial index. This is clearly not the case, again indicating that the results do not support the DCC with a multivariate Student t distribution. Moreover, in 20% of all cases the copula that we select presents some kind of asymmetry. Most of this asymmetry can be found in the dependence of stocks and sectors with the $S\&P500$ index return. In 30% of the cases, the copulas for the dependence of the individual stocks or sectors with the market are asymmetric as can be seen in the left panel of Table 4, and the proportion is much lower in the case of sectorial dependence, conditional on the market. Not all sectors present the same kind of asymmetry in the dependence, though, and we can break up sectors into roughly three groups depending on the number of asymmetric copulas that we select for the stocks in each sector. The low asymmetry group contains *Financials* (1/0)[11], *IT* (0/0) and *Telecom* (0/1), since there are almost no stocks in these sectors that present asymmetry. The second group contain sectors with low to average asymmetry. This group contains sectors where the percentage of asymmetric copulas is equal to or lower than 20%, and its members

[11] (x/y): x is the number of asymmetric copulas selected between the stocks of the specific sector and the $S\&P500$ index return. y is the number of asymmetric copulas in the case of sectorial dependence, conditional on the market.

	Dependence with the market							Dependence with the own sector given the market						
	Model	τ/ω	Dof	τ	Prob	α	β	Model	τ/ω	Dof	τ	Prob	α	β
ENERGY	RGumbel	0.31												
XOM	RGumbel	0.44				0.04	0.93	Gaussian	0.63					
CVX	Gaussian	0.27						Gumbel RGumbel	0.54		0.61	0.50		
COP	Gaussian	0.24						Student t	0.80	7.40			0.02	0.97
SLB	RGumbel	0.25						Frank	0.51					
OXY	Gaussian	0.25						Gaussian	0.44				0.03	0.97
ESV	RGumbel	0.18						Gaussian	0.44					
BJS	Student t	0.20	7.86					Gaussian	0.42					
SUN	RGumbel	0.23						Frank	0.51				0.04	0.95
RDC	RGumbel	0.22						Frank	0.48					
TSO	Clayton	0.17						Frank	0.24				0.05	0.94
INDUST	RGumbel	0.86				0.05	0.93							
GE	Student t	0.49	6.77					Gaussian	0.61				0.04	0.91
UTX	Student t	0.40	12.54					Student t	0.26	8.77				
BA	Gaussian	0.51				0.04	0.93	Gaussian	0.24					
MMM	Frank	0.34						Gumbel	0.23					
CAT	Frank	0.61				0.05	0.92	Gaussian	0.25					
PLL	RGumbel	0.44				0.05	0.91	Frank	0.11					
AW	Frank	0.13				0.02	0.98	RGumbel	0.05					
R	RGumbel	0.32						Gumbel	0.15					
CTAS	Student t	0.33	7.51			0.02	0.98	Frank	0.12					
RHI	Frank	0.59				0.05	0.91	Frank	0.07					
HEALTH	RGumbel	0.66				0.08	0.88							
JNJ	RGumbel	0.37				0.05	0.88	Gaussian	0.40					
PFE	Frank	0.51				0.07	0.85	Gumbel RGumbel	0.42		0.56	0.69		
MRK	Frank	0.52				0.07	0.87	Student t	0.42	11.56				
ABT	RGumbel	0.37				0.05	0.89	Gaussian	0.36					
WYE	Frank RGumbel	0.18		0.57	0.73			Gaussian	0.35					
MYL	RGumbel	0.15						Gumbel	0.09					
MIL	Frank	0.31						Independent						
PKI	Frank	0.31						Independent						
WPI	RGumbel	0.20						Gumbel	0.08					
THC	RGumbel	0.18						Frank	0.15					
FINAN	Student t	0.87	11.40			0.06	0.90							
BAC	Gaussian	0.45						Gaussian Frank	0.59		0.32	0.50		
JPM	Frank RGumbel	0.57		0.45	0.41			Student t	0.40	8.07				
C	Gaussian	0.52						Student t	0.37	13.20				
AIG	Student t	0.64	9.08			0.05	0.87	Gaussian	0.34					
WFC	Gaussian	0.42						Student t	0.39	11.11				
HBAN	Gaussian	0.54				0.05	0.92	Student t	0.19	8.54			0.02	0.98
SOV	Gaussian	0.33						Gaussian	0.14					
MBI	Frank	0.57				0.06	0.88	Gaussian	0.24					
FHN	Frank	0.35						Gaussian	0.31					
MTG	Frank	0.55				0.04	0.93	Gaussian	0.24					
UTIL	Gaussian	0.49				0.02	0.97							
EXC	Gaussian	0.30				0.03	0.96	Student t	0.77	5.34			0.02	0.97
SO	RGumbel	0.01				0.02	0.98	Student t	0.51	6.72				
FPL	Gaussian	0.34				0.03	0.96	Student t	0.71	7.69			0.04	0.86
D	Gaussian	0.34				0.02	0.96	Gaussian Frank	0.59		0.35	0.68		
DUK	Gaussian	0.38				0.03	0.96	Gaussian	0.52					
TEG	RGumbel	0.21						Gaussian	0.41					
TE	Gaussian	0.39				0.02	0.97	Student t	0.46	8.98				
PNW	Gaussian	0.32				0.03	0.96	Gaussian	0.45					
CMS	Frank	0.45				0.02	0.97	Gaussian	0.36					
GAS	Gaussian	0.39				0.02	0.97	Gaussian	0.32					

	Dependence with the market							Dependence with the own sector given the market						
	Model	τ/ω	Dof	τ	Prob	α	β	Model	τ/ω	Dof	τ	Prob	α	β
MATERIALS	Gaussian	0.70				0.07	0.90							
DD	Gaussian	0.58				0.04	0.93	Gaussian	0.53				0.02	0.98
DOW	Gaussian	0.53				0.04	0.94	Gaussian	0.60				0.03	0.96
AA	Gaussian	0.45				0.05	0.94	Student t	0.43	7.72				
PX	RGumbel	0.33				0.02	0.98	Gaussian	0.31				0.02	0.97
NUE	RGumbel	0.20				0.01	0.99	Frank	0.34					
SEE	Frank RGumbel	0.61		0.12	0.42			Gaussian	0.16					
IFF	Frank	0.32						Gaussian	0.11					
ASH	RGumbel	0.30						Frank	0.15					
BMS	Gaussian Frank	0.56		0.22	0.43			Frank	0.08				0.02	0.98
HPC	Frank	0.30						Frank	0.29					
C DISCR	Gaussian	0.78				0.02	0.98							
MCD	Gaussian	0.30						Frank	0.14					
CMCSA	RGumbel	0.30						Frank	0.12					
DIS	Student t	0.55	7.03			0.03	0.93	Frank	0.22					
TWX	RGumbel	0.47				0.02	0.97	Frank	0.09					
HD	Frank RGumbel	0.56		0.32	0.36			Gaussian	0.36					
KBH	RGumbel	0.33						Gaussian	0.12					
LIZ	Frank	0.30						Gaussian	0.20					
JNY	Frank	0.30						Gaussian	0.21					
MDP	Frank RGumbel	0.14		0.45	0.40			Frank	0.13					
DDS	Frank	0.31						Student t	0.22	9.77				
C STAP	Student t	0.67	6.48			0.05	0.92							
PG	Student t	0.22	7.31					Student t	0.43	9.69				
WMT	Student t	0.32	11.87					Frank	0.13					
KO	Gaussian	0.44				0.10	0.83	Student t	0.44	9.99				
PEP	Student t	0.24	6.45					Gaussian	0.56				0.02	0.97
CVS	Gaussian	0.23						Gaussian	-0.07				0.01	0.99
BF.B	RGumbel	0.40				0.05	0.92	Student t	0.20	8.68				
TSN	Frank	0.19						Gumbel	0.07					
WFMI	Frank	0.37				0.07	0.82	Independent						
MKC	RGumbel	0.13						Gumbel	0.12					
STZ	Gaussian	0.16						Independent						
IT	Gaussian	0.56				0.02	0.98							
MSFT	Frank	0.41						Gaussian	0.52				0.03	0.96
IBM	Frank	0.39						Gaussian	0.29					
AAPL	Frank	0.24						Gaussian	0.22					
CSCO	Frank	0.42						Student t	0.45	11.88				
INTC	Frank	0.40						Gaussian Frank	0.34		0.68	0.65		
TLAB	Frank	0.36						Frank	0.21					
NOVL	Gaussian	0.29						Frank	0.13					
CPWR	Frank	0.31						Frank	0.16					
QLGC	Frank	0.29						Gaussian	0.07				0.02	0.98
UIS	Frank	0.32						Frank	0.17					
TELECOM	Gaussian	0.45												
T	Gaussian	0.46				0.03	0.95	Student t	0.66	7.62			0.02	0.98
VZ	Frank	0.34						Student t	0.51	6.86				
S	Frank	0.32						Gaussian	0.29					
CTL	Gaussian	0.30						Frank	0.19					
CZN	Frank	0.24						Clayton	0.11					

Table 4: Copula results

The half left part contain the results of the dependence models of each individual stock and sector with the S&P500 market return. The half right part of the table contains the results of the dependence between individual stock returns and the sectors given the market. "Dof" contains the degrees of freedom for the Student t copula. "τ/ω" refers to the Kendall's tau in the case of constant models and to the constant in the time-varying models. "τ" and "Prob" respectively refer to the Kendall's tau of the second copula in a mixture model and the weight of the first copula component in the mixture. "α" and "β" are the autoregressive coefficients of the dynamic copulas.

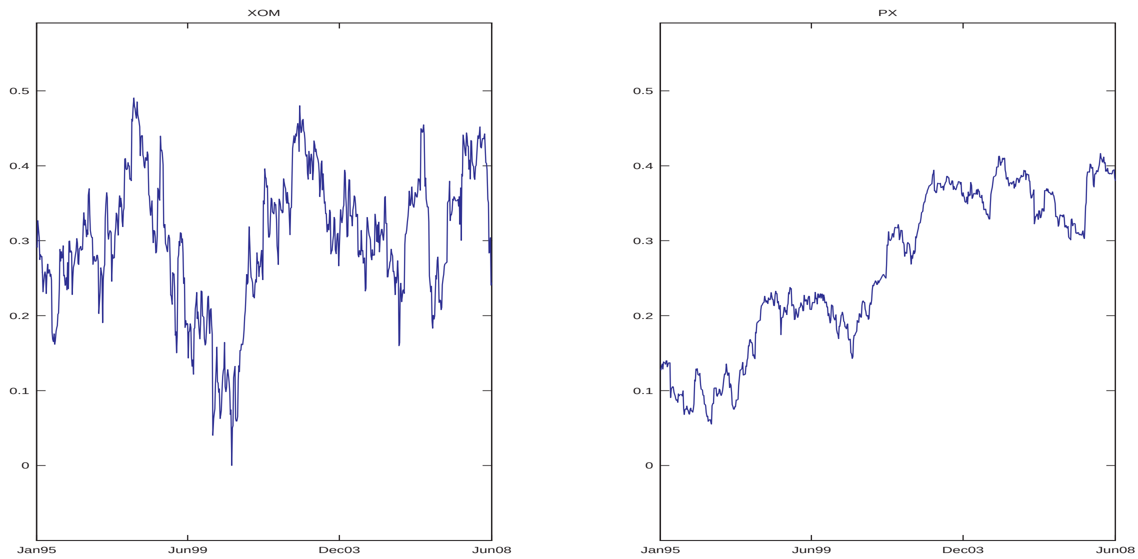

Figure 2: Time varying Kendall's tau for two stocks with the market
This figure shows the time-varying Kendall's tau implied by the estimated rotated Gumbel copulas of XOM (*Energy*) and PX (*Materials*) with the market.

are *Utilities* (2/0), *Materials* (4/0) and *Consumer Staples* (2/2). The last group contains sectors with more prevalent asymmetry and contains *Energy* (6/1), *Industrials* (2/3), *Health* (6/2) and *Consumer Discretionary* (5/0). In the case of the *Energy*, *Industrials* and *Health* sectors, in addition to the asymmetry between the stocks and the market return, there is also asymmetry between the sectorial returns and the market. If we focus on the type of dynamics, it is striking that most of the copulas that are selected with the BIC are constant, and we select time-varying copulas only in 30% of all cases.

As far as time varying dependence is concerned, most of the selected time varying copulas are found for the dependence of stocks and sectors with the $S\&P500$ index return. Forty-three (out of 105) of the copulas for the dependence of the individual stocks or sectors with the market are time varying. Moreover these 43 cases represent 72% of all selected time varying copulas. Another important feature is that the estimated autoregressive parameters are not equal across stocks in the same sector, nor across stocks with the same type of copula. However, there is a tendency for stocks in the same sector with time varying copula to present similar autoregressive parameters. Moreover, we find that in 13 cases the sum of the autoregressive parameters are very close to 1. This can be due to structural breaks in the dependence parameter, but plotting the time-varying dependence parameter over the sample period reveals that in most cases the dependence is smoothly trending upwards. Figure 2 plots the dynamic Kendall's tau implied by the estimated rotated Gumbel copulas of XOM (*Energy*) and PX (*Materials*) with the market. While XOM (left panel) shows mean-reverting dependence, PX (right panel) is close to being integrated and shows an upward trend. The dynamic structure we find seems incompatible with the DCC, which implies that the autoregressive parameters are the same across all stocks.

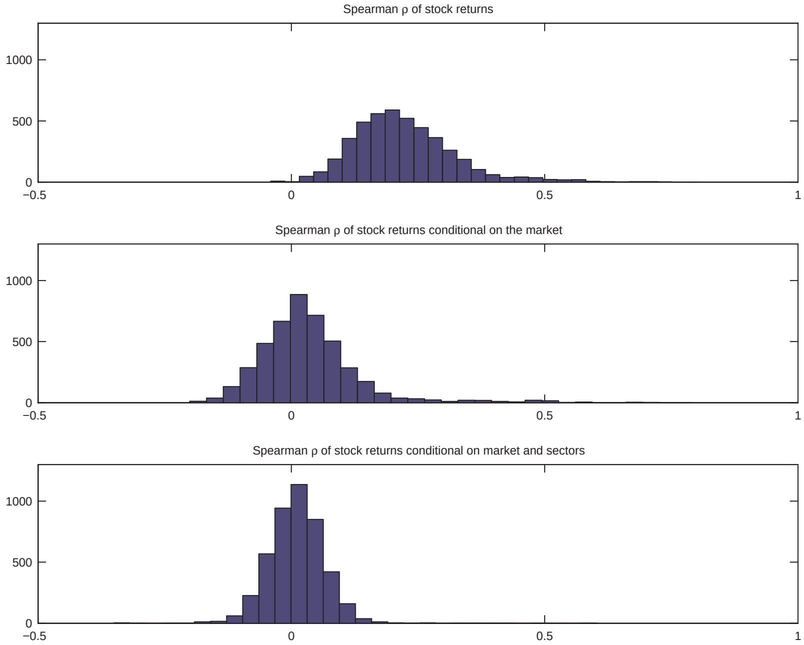

Figure 3: Histogram of Spearman rank correlation of the 95 S&P500 stock returns
The three plots show the histograms of the Spearman rank correlation for all pair of assets returns. The top graph is computed on the returns after the marginal models have been estimated. The middle graph shows the Spearman correlation after the dependence with the market has been taken into account. Finally the bottom graph shows the rank correlations after the effect of the dependence with the market and the sectors have been taken out.

Figure 3 and Table 5 show how the overall level of dependence between all pairs of stocks is modified by conditioning on the $S\&P500$ and the sectorial returns. Figure 3 shows histograms of the empirical Spearman rank correlation for all pair of assets returns. From top to bottom, it shows the Spearman rank correlation of the returns, after the marginal models have been estimated (top panel), after the dependence with the market has been taken into account (middle panel), and after the effect of the dependence with the market and the sectors have been taken out (bottom panel).[12] Table 5 contains some descriptive statistics for the distributions that we plot in Figure 3. The histograms show how the Spearman rank correlations get more and more concentrated around zero after conditioning on the market ($S\&P500$). The average Spearman rank correlation is reduced from 0.22 to 0.03, while there is no change in the standard deviation. With absolute values, the average dependence is still reduced very significantly from 0.22 to 0.07 and there is also a decrease in the standard deviation, from 0.10 to 0.08. After conditioning on the market and the sectors, the average Spearman rank correlation gets even closer to zero, from 0.03 to 0.01

[12] The top graph shows the empirical Spearman rank correlation of $F(r_1^{S_1}), F(r_2^{S_1}), \cdots F(r_I^{S_J})$, where $F(\cdot)$ is the marginal model for each return. The middle graph shows the empirical Spearman rank correlation of $F(r_1^{S_1}|r_M), F(r_2^{S_1}|r_M), \cdots, F(r_I^{S_J}|r_M)$, where $F(\cdot|r_M)$ is the marginal model for each return given the market. Finally the bottom graph shows the empirical Spearman rank correlation of $F(r_1^{S_1}|r_M, r_{S_1}, \cdots, r_{S_J}), F(r_2^{S_1}|r_M, r_{S_1}, \cdots, r_{S_J}), \cdots, F(r_I^{S_J}|r_M, r_{S_1}, \cdots, r_{S_J})$, where $F(\cdot|r_M, r_{S_1}, \cdots, r_{S_J})$ is the marginal model for each return given the market and all sectors.

	Mean	Std dev	Max	Min	Mean abs. value	Std dev abs. value
ρ_S	0.22	0.10	0.81	-0.04	0.22	0.10
ρ_S condtional on market	0.03	0.10	0.79	-0.20	0.07	0.08
ρ_S cond. on market and sector	0.01	0.06	0.60	-0.35	0.04	0.04

Table 5: Descriptive statistics of Spearman rank correlation of stock returns

The first line is computed with Spearman rank correlations after GARCH effects have been taken out. The second line is the same, but after the dependence with the market has also been taken out. Finally, the third line is computed after GARCH effects and dependence with market and sectors have been taken into account.

but the main change is the sharp reduction in standard deviation, from 0.10 to 0.06. The same behavior is present in the case of the absolute value of the Spearman rank correlation: the average is reduced from 0.07 to 0.04, and the standard deviation decreases from 0.08 to 0.04. These results mean that most of the dependence between stocks can be captured by the $S\&P500$ and the sectorial returns, since, after conditioning on the market and sectors, only 33% of all possible pairs have an absolute Spearman rank correlation higher than 0.05 and this percentage is reduced to 6% when we consider an absolute Spearman rank correlation of 0.10. If the CAPM model holds, then even after controlling only linearly for the market, the idiosyncratic error terms should all be independent. However, we want to account for the fact that even though the remaining dependence is generally small, it is not exactly zero. In order to be sure that we capture the remaining dependence we estimate a multivariate Gaussian copula for the dependence between stock returns conditional on market and sector returns. A likelihood ratio test strongly rejects the null hypothesis of independence for the Gaussian copula parameter with a test statistic of 8744.2 for a 5% critical value of $\chi^2_{[95.94/2]} = 4621.3$, which implies a p-value of 0.00.

To summarize, our results confirm that many of the restrictions imposed on the data by the DCC can be very limiting. First, it appears clearly that the marginal distributions of the returns are not Gaussian and moreover the dependence with the market is not Gaussian, as is implied by the DCC. In 30% of the cases we select an asymmetric copula for the dependence with the market. Also, the DCC imposes that the persistence in the dependence between all pairs of assets is the same. Again our results demonstrate that this is not the case. For many assets, we find that the dependence between the market and the stocks does not vary over time. For some stocks, we find on the contrary that the sum of the autoregressive parameters is one, implying that the dynamics of the rank correlation is integrated. This shows the difficulties that arise if one restricts the dynamics to be the same for all pairs of asset, as is done in the DCC. Finally, we are able to capture nearly all the dependence present in the data, just by conditioning on the market and the sector with a dynamic and flexible copula model. For the remaining dependence we use a multivariate Gaussian copula.

5.3 Evaluation of VaR

We evaluate the performance of our model[13] in terms of its ability to generate good estimates of Value at Risk. We use two criteria to judge performance. The first one is the likelihood ratio test of correct unconditional coverage of Kupiec (1995), based on the binomial distribution. The comparison of the VaR forecast and the realized returns of the portfolio defines a sequence of binary variables, I_t, called hits. A hit occurs whenever the observed return is lower than the predicted VaR, resulting in a violation of the threshold. A successful VaR model at a threshold α should have close to a fraction α of violations in sample. Kupiec (1995) proposes to test $H_0 : f = \alpha$ against $H_1 : f \neq \alpha$, where f is the failure rate (estimated by $\hat{f}$, the empirical failure rate). Under the null hypothesis, the Kupiec likelihood ratio statistic $LR = 2ln[\hat{f}^N(1-\hat{f})^{T-N}] - 2ln[\alpha^N(1-\alpha)^{T-N}]$ is asymptotically distributed as a $\chi^2(1)$, where N is the number of VaR violations, T is the total number of observations (703 in our case) and α is the failure rate of the null hypothesis. The second criterion is the likelihood ratio test of conditional coverage of Christoffersen (1998). This test is a combination of the unconditional coverage and the independence tests. For the independence test, Christoffersen (1998) considers a binary first-order Markov chain for the hits, with transition probability matrix Π_1

$$\Pi_1 = \begin{bmatrix} 1-\pi_{01} & \pi_{01} \\ 1-\pi_{11} & \pi_{11} \end{bmatrix},$$

where $\pi_{i,j} = Pr(I_t = j | I_{t-1} = i)$. Under the null hypothesis, $H_0 : \pi_{01} = \pi_{11} = \alpha$ the likelihood ratio test of conditional coverage,

$$LR = 2ln[(1-\hat{\pi_{01}})^{n_{00}}\hat{\pi_{01}}^{n_{01}}(1-\hat{\pi_{11}})^{n_{10}}\hat{\pi_{11}}^{n_{11}}] - 2ln[\alpha^N(1-\alpha)^{T-N}],$$

is asymptotically distributed as a $\chi^2(2)$, where n_{ij} is the number of hits with value i followed by j, $\hat{\pi_{01}} = n_{01}/(n_{00}+n_{01})$ and $\hat{\pi_{11}} = n_{11}/(n_{10}+n_{11})$.

We use the DCC model as a benchmark for the evaluation of our model. The problem with the DCC in large dimensions is that it requires inversion of a large correlation matrix at each period and that its parameters suffer from a downward bias, as explained in Engle et al. (2008). Instead of attempting a direct estimation of the DCC model for the whole vector of returns, we rely on the profile likelihood approach of Engle et al. (2008) and estimate the parameters that maximize the sum of likelihoods of bivariate DCC models of 50 pairs, chosen randomly from the whole cross-section of the data. Even though our model is suitable for very large dimensional problems, we concentrate on portfolios of eight stocks, four big and four small, taken from two sectors, *Health* and *Consumer Staples*. They are, in decreasing order of market capitalization, JNJ, PFE, WPI, THC for the *Health* sector and PG, WMT, MKC and STZ for the *Consumer Staples* sector. We consider six different portfolios whose weights are shown in Table 6. We compute the upper and lower tail percentiles of these portfolios, which corresponds to analyzing the risk of, respectively, long and short positions in the portfolios of Table 6. All models are tested with a VaR[14] level

[13]We consider only the market sector model, based on the structure of Equation (7) which, for the rest of this section we refer to as the CAVA model.

[14]The VaR for the CAVA model has been calculated by simulating 50.000 observations, following the algorithm of Section 7.4.

Portfolio	Portfolio weights							
	Sector 1				Sector 2			
	Big		Small		Big		Small	
1: Long all stocks	1/8	1/8	1/8	1/8	1/8	1/8	1/8	1/8
2: Long sector 1	1/4	1/4	1/4	1/4	0	0	0	0
3: Long sector 2	0	0	0	0	1/4	1/4	1/4	1/4
4: Long big stocks, short small stocks	1/8	1/8	-1/8	-1/8	1/8	1/8	-1/8	-1/8
5: Long big stocks	1/4	1/4	0	0	1/4	1/4	0	0
6: Long small stocks	0	0	1/4	1/4	0	0	1/4	1/4

Table 6: Portfolios for Value at Risk computation

This table shows the weights of the 6 portfolios we use to evaluate Value at Risk.

α of 10%, 5%, 2.5% and 1%, and their performance is assessed by likelihood ratio tests of correct unconditional and conditional coverage, shown in Table 7.

To compare the results of the models, we use a summary performance measure, Grade(β), which is the percentage of p-values above the β critical value for long and short positions. The level of the test β should not be confused with the level α at which VaR is evaluated. For instance, for the correct unconditional coverage test, in Table 7, when we consider long positions at the critical value $\beta = 5\%$, we have 5 rejections out of 24 portfolios for the DCC, which implies that its $Grade(5\%)$ is 79%, compared to 100% for the CAVA model. If instead we consider a critical value of $\beta = 10\%$, we have 6 rejections for the DCC and 1 for the CAVA model, which implies $Grade(10\%)$ of 75% for the DCC and 92% for the CAVA. Overall the CAVA model performs better than the DCC model for both the conditional and the unconditional coverage tests. We repeat this analysis with another set of stocks: XOM, OXY, ESV, TSO for the *Energy* sector and GE, CAT, PLL and RHI for the *Industrials* sector, given in decreasing order of market capitalization. We consider the same portfolios as in the previous exercise and a summary of the results is shown in Table 8. Overall the results in terms of $Grade(5\%)$ and $Grade(10\%)$ are quite similar to the ones in Table 7. The CAVA performs better than the DCC and the main differences are in short trading positions.

So far we have evaluated VaR in-sample and this exercise can be viewed as providing diagnostics for the model. Given that the CAVA is more heavily parameterized than the DCC, it is of interest to check that its good in-sample performance carries over to an out-of-sample setting. In order to do this, we split the sample into an estimation period of 500 observations and leave 203 observations for the out-of-sample evaluation. We specify and estimate the models[15] on the first 500 observations and perform a series of 203 one-step ahead VaR forecasts. Keeping the structure constant, we reestimate the parameters of each

[15] This includes the type of GARCH models for both the DCC and the CAVA, as well as the bivariate copulas for the CAVA model.

		Unconditional Coverage				Conditional Coverage			
		Long positions		Short positions		Long positions		Short positions	
	α	CAVA	DCC	CAVA	DCC	CAVA	DCC	CAVA	DCC
Long all	10%	0.07*	0.92	0.59	0.02**	0.05*	0.08*	0.75	0.07*
	5%	0.19	0.25	0.59	0.03**	0.16	0.17	0.78	0.04**
	2.5%	0.30	0.09*	0.91	0.37	0.55	0.14	0.77	0.51
	1%	0.29	0.01**	0.72	0.69	0.18	0.02**	0.85	0.88
Long sector 1	10%	0.73	0.78	0.51	0.05**	0.58	0.56	0.10*	0.06*
	5%	0.88	0.74	0.59	0.02**	0.34	0.34	0.19	0.03**
	2.5%	0.30	0.02**	0.70	0.09*	0.29	0.05*	0.64	0.20
	1%	0.29	0.00**	0.99	0.99	0.49	0.00**	0.93	0.93
Long sector 2	10%	0.59	0.11	0.55	0.19	0.86	0.27	0.08*	0.42
	5%	0.71	0.71	0.99	0.71	0.51	0.88	0.17	0.19
	2.5%	0.42	0.56	0.53	0.37	0.65	0.73	0.59	0.52
	1%	0.47	0.16	0.21	0.99	0.69	0.14	0.45	0.93
Long big, short small	10%	0.43	0.01**	0.55	0.51	0.68	0.02**	0.59	0.80
	5%	0.99	0.28	0.51	0.88	0.81	0.16	0.80	0.77
	2.5%	0.42	0.53	0.14	0.21	0.38	0.59	0.33	0.44
	1%	0.72	0.99	0.47	0.29	0.85	0.93	0.69	0.18
Long big	10%	0.82	0.24	0.29	0.06*	0.95	0.49	0.26	0.07*
	5%	0.59	0.28	0.99	0.59	0.15	0.05*	0.16	0.19
	2.5%	0.89	0.89	0.30	0.73	0.65	0.65	0.29	0.56
	1%	0.47	0.29	0.47	0.29	0.69	0.49	0.69	0.49
Long small	10%	0.15	0.73	0.59	0.11	0.31	0.40	0.75	0.29
	5%	0.07*	0.25	0.59	0.10*	0.02**	0.01**	0.43	0.05*
	2.5%	0.91	0.21	0.89	0.25	0.21	0.21	0.72	0.25
	1%	0.99	0.04**	0.72	0.21	0.16	0.01**	0.85	0.45
Grade(5%)		100%	79%	100%	83%	96%	75%	100%	92%
Grade(10%)		92%	75%	100%	71%	92%	67%	92%	75%

Table 7: Unconditional and conditional coverage tests for in-sample weekly VaR

This table shows p-values of the Kupiec test for unconditional coverage (left panel) and of the Christoffersen test of conditional coverage (right panel) for the portfolios described in Table 6. The portfolios are composed of eight stocks, four big and four small, taken from two sectors, *Health* and *Consumer Staples*. They are, in decreasing order of market capitalization, JNJ, PFE, WPI, THC for the *Health* sector and PG, WMT, MKC and STZ for the *Consumer Staples* sector. One star, respectively two, indicates that we reject the null at the 10%, respectively 5% level of significance. Grade(β), is the percentage of p-values above the β critical value for long and short positions. The level of the test β should not be confused with the level α at which VaR is evaluated.

model every 20 periods, using an expanding window. We evaluate VaR using the same portfolios and the same two sets of stocks that we use in-sample and the results are given in Table 9. The results show that the CAVA outperforms the DCC in all cases. Overall we reject the null of correct coverage more often than in the in-sample exercise, which is a reflection of the difficulty of forecasting. The difference between both models are smaller than in the in-sample exercise, which can be due to the fact that the sample size is much

	Unconditional Coverage				Conditional Coverage			
	Long positions		Short positions		Long positions		Short positions	
	CAVA	DCC	CAVA	DCC	CAVA	DCC	CAVA	DCC
					Energy - Industrials			
Grade(5%)	100%	100%	100%	75%	100%	100%	96%	96%
Grade(10%)	100%	75%	87%	58%	100%	87%	96%	71%

Table 8: Summary results of in-sample weekly VaR for an alternative set of stocks

This table shows summary results of the Kupiec test for unconditional coverage (left panel) and of the Christoffersen test of conditional coverage (right panel) for an alternative set of stocks: XOM, OXY, ESV, TSO for the *Energy* sector and GE, CAT, PLL and RHI for the *Industrials* sector, given in decreasing order of market capitalization. We consider long and short positions in the same 6 portfolios presented in Table 6. The table presents Grade(β), a performance measure, which reports the percentage of p-values above the β critical value for long and short positions for the correct coverage test and the conditional coverage test.

smaller. There is however a pattern in that the CAVA offers higher improvements over the DCC in short positions than in long positions.

6 Conclusion

This paper introduces the canonical vine autoregressive (CAVA) model, a new multivariate GARCH model which does not suffer from the limitations of the existing models. It can easily be used for high-dimensional cross-sections of assets while allowing for empirically relevant departures from elliptical distributions, like skewness, kurtosis and asymmetric dependence. Using the concept of copulas, we split the joint distribution function of the returns into two parts, one for the marginals and one that captures the multivariate dependence with a series of dynamic bivariate copulas. The idea of breaking the joint distribution into marginals and dependence is not new, as it underlies the DCC model, which separates the dynamics of the volatility of each asset and the conditional correlation between all assets. However, unlike the DCC, the CAVA does not restrict the marginal GARCH models to be Gaussian. In order to model dependence we use a canonical vine copula, which was recently introduced into the financial literature by Aas et al. (2009), and which accommodates very flexible types of dependence. It is based on decomposing iteratively a multivariate copula into a product of bivariate conditional copulas, each of which can be chosen from a long list, producing a large, flexible class of models. Moreover, as a fully general decomposition of the canonical vine model for really large dimensions would not be sensible, we follow financial practice and focus on the dependence between stocks and the market return. In that sense our model can be viewed as a time-varying non-Gaussian extension of the CAPM. We use weekly returns on 95 S&P500 stocks taken from 10 different sectors. A first model considers the dependence between all stocks and the market and uses a Gaussian copula for the remaining dependence between the stocks, conditional on the market. A second model considers a market sector model, where after conditioning on the market we consider the dependence with the sectorial returns. Our results confirm

	Unconditional Coverage				Conditional Coverage			
	Long positions		Short positions		Long positions		Short positions	
	CAVA	DCC	CAVA	DCC	CAVA	DCC	CAVA	DCC
					Health - Consumer Staples			
$Grade(5\%)$	96%	92%	83%	67%	100%	92%	92%	83%
$Grade(10\%)$	92%	88%	67%	54%	88%	83%	83%	71%
					Energy - Industrials			
$Grade(5\%)$	88%	88%	58%	50%	96%	96%	79%	75%
$Grade(10\%)$	88%	79%	54%	46%	96%	88%	67%	63%

Table 9: Summary results of out-of-sample weekly VaR

This table shows summary results of the Kupiec test for unconditional coverage (left panel) and of the Christoffersen test of conditional coverage (right panel) for two sets of eight stocks. The first set is composed of four big and four small stocks, taken from two sectors, *Health* and *Consumer Staples*. They are, in decreasing order of market capitalization, JNJ, PFE, WPI, THC for the *Health* sector and PG, WMT, MKC and STZ for the *Consumer Staples* sector. The second set is composed of XOM, OXY, ESV, TSO belonging to the *Energy* sector and GE, CAT, PLL and RHI belonging to the *Industrials* sector, given in decreasing order of market capitalization. We consider long and short positions in the same 6 portfolios presented in Table 6. The table presents Grade(β), a performance measure, which reports the percentage of p-values above the β critical value for long and short positions for the correct coverage test and the conditional coverage test.

that many of the restrictions imposed by the DCC can be very limiting. First it appears clearly that neither the marginal distribution nor the dependence structure is Gaussian, as is implied by the DCC. Second, the DCC imposes that the persistence in the dependence between all pairs of assets is the same. Again our results demonstrate that this is not the case. In terms of in-sample and out-of-sample VaR we show that the CAVA performs better than the DCC in all portfolios that we consider.

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7 Supplemental Materials

7.1 GARCH Models

GARCH

$$h_t = \omega + \alpha\epsilon_{t-1}^2 + \beta h_{t-1}$$

GJR-GARCH

$$h_t = \omega + \alpha\epsilon_{t-1}^2 + \gamma \mathbf{1}_{[\epsilon_{t-1}<0]}\epsilon_{t-1}^2 + \beta h_{t-1}$$

AVGARCH

$$h_t^{1/2} = \omega + \alpha|\epsilon_{t-1}| + \beta h_{t-1}^{1/2}$$

ZARCH

$$h_t^{1/2} = \omega + \alpha|\epsilon_{t-1}| + \gamma \mathbf{1}_{[\epsilon_{t-1}<0]}|\epsilon_{t-1}| + \beta h_{t-1}^{1/2}$$

EGARCH

$$log(h_t) = \omega + \alpha \left(\frac{|\epsilon_{t-1}|}{\sqrt{h_{t-1}}} - \mathbb{E}\left|\frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}}\right| \right) + \gamma \frac{\epsilon_{t-1}}{\sqrt{h_{t-1}}} + \beta log(h_{t-1})$$

APARCH

$$h_t^{\delta/2} = \omega + \alpha \left(|\epsilon_{t-1}| + \gamma\epsilon_{t-1}\right)^{\delta} + \beta h_{t-1}^{\delta/2}$$

NARCH

$$h_t^{\delta/2} = \omega + \alpha|\epsilon_{t-1}|^{\delta} + \beta h_{t-1}^{\delta/2}$$

7.2 Copulas

7.2.1 Gaussian Copula

The distribution function of the Gaussian copula is:

$$C_N(u_1, \ldots, u_n; \Sigma) = \Phi_\Sigma(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)),$$

where Φ^{-1} denotes the inverse cumulative density of the standard normal and $\Phi_\Sigma(x_1, \ldots, x_n; \Sigma)$ denotes the standard multivariate normal cumulative distribution:

$$\Phi_\Sigma(x_1, \ldots, x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{1}{(2\pi)^{n/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}v'\Sigma^{-1}v\right) dv,$$

where $v = (v_1, \ldots, v_n)$ and Σ is a correlation matrix, that is symmetric, semi-definite positive with ones on the diagonal and off-diagonal terms between -1 and 1. The corresponding density is:

$$c_N(u_1, \ldots, u_n; \Sigma) = |\Sigma|^{-1/2} exp\left[-\frac{1}{2}\left(x'\Sigma^{-1}x - x'x\right)\right],$$

where $x = (\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n))$. The bivariate version that we use in the canonical vine copulas is:

$$c_N(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left[\frac{-[\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2 - 2\rho\Phi^{-1}(u_1)\Phi^{-1}(u_2)]}{2(1-\rho^2)} + \frac{\Phi^{-1}(u_1)^2 + \Phi^{-1}(u_2)^2}{2}\right],$$

Define the conditional copula $h(u_1, u_2, \theta) = F(u_1|u_2) = \frac{\partial C_{u_1,u_2}(u_1,u_2,\theta)}{\partial u_2}$. For the normal copula, it is given by

$$h_N(u_1, u_2, \rho_{12}) = \Phi\left(\frac{\Phi^{-1}(u_1) - \rho_{12}\Phi^{-1}(u_2)}{\sqrt{1-\rho_{12}^2}}\right),$$

and its inverse for a given value of the conditioning variable is:

$$h_N^{-1}(u_1, u_2, \rho_{1,2}) = \Phi\left(\Phi^{-1}(u_1)\sqrt{1-\rho_{12}^2} + \rho_{1,2}\Phi^{-1}(u_2)\right),$$

where ρ is a correlation coefficient that lies between -1 and 1.
The Gaussian copula has zero upper and lower tail dependence, $\lambda_U = \lambda_L = 0$, except in the case of perfect correlation, $\rho = 1$.

7.2.2 Multivariate Student t Copula

The distribution function of the Student t copula is:

$$C_T(u_1, \ldots, u_n; \Sigma, \nu) = T_{\Sigma,\nu}(T_\nu^{-1}(u_1), \ldots, T_\nu^{-1}(u_n)),$$

where $T_\nu^{-1}(v)$ is the inverse of the cumulative distribution function of the univariate Student t with ν degrees of freedom and $T_{\Sigma,\nu}$ is given by:

$$T_{\Sigma,\nu}(x_1, \ldots, x_n;) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^n|\Sigma|}} \left(1 + \frac{v'\Sigma^{-1}v}{\nu}\right)^{\frac{-\nu+n}{2}} dv,$$

where $v = (v_1, \ldots, v_n)$ and Σ is a correlation matrix, that is symmetric, semi-definite positive with ones on the diagonal and off-diagonal terms between -1 and 1. The corresponding density is:

$$c_T(u_1, \ldots, u_n; \Sigma, \nu) = \frac{\Gamma\left(\frac{\nu+n}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^n|\Sigma|}} \frac{1}{\prod_{i=1}^n f_\nu(T_\nu^{-1}(u_i))} \left(1 + \frac{x'\Sigma^{-1}x}{\nu}\right)^{\frac{-\nu+n}{2}},$$

where $x = (T_\nu^{-1}(u_1), \ldots, T_\nu^{-1}(u_n))$ and $f_\nu(.)$ is the density of the Student t distribution with ν degrees of freedom, $\rho \in (-1, 1)$ and $\nu > 2$. The bivariate version that we use in the canonical vine copulas is:

$$c_T(u_1,u_2;\rho,\nu)=\Gamma\left(\frac{\nu+2}{2}\right)\frac{1+\left(\frac{T_\nu^{-1}(u_1)^2+T_\nu^{-1}(u_2)^2-2\rho T_\nu^{-1}(u_1)T_\nu^{-1}(u_2)}{\nu(1-\rho^2)}\right)^{-\left(\frac{\nu+2}{2}\right)}}{f_\nu(T_\nu^{-1}(u_1))f_\nu(T_\nu^{-1}(u_2))\nu\Pi\Gamma(\frac{\nu}{2})\sqrt{1-\rho^2}}.$$

The conditional copula is given by:

$$h_T(u_1,u_2,\rho_{12},\nu)=T_{\nu+1}\left(\frac{T_\nu^{-1}(u_1)-\rho_{12}T_\nu^{-1}(u_2)}{\sqrt{\frac{(\nu+(T_\nu^{-1}(u_2))^2)(1-\rho_{12})}{\nu+1}}}\right),$$

and its inverse for a given value of the conditioning variable is:

$$h_T^{-1}(u_1,u_2,\rho_{12},\nu)=T_\nu\left(T_{\nu+1}^{-1}(u_1)\sqrt{\frac{(\nu+(T_\nu^{-1}(u_2))^2)(1-\rho_{1,2})}{\nu+1}}+\rho_{12}T_\nu^{-1}(u_2)\right).$$

The Student t copula has the same lower and upper tail dependence for every pair of variables: $\lambda_U=\lambda_L=2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$.

7.2.3 Bivariate Gumbel and Rotated Gumbel Copula

The Gumbel copula has the following distribution:

$$C_G(u_1,u_2,\theta)=exp\left(-((-\log u_1)^\theta+(-\log u_2)^\theta)^{1/\theta}\right),$$

and the following density:

$$c_G(u_1,u_2,\theta)=\frac{C_G(u_1,u_2,\theta)(\log u_1.\log u_2)^{\theta-1}}{u_1u_2((-\log u_1)^\theta+(-\log u_2)^\theta)^{2-1/\theta}}\left(((-\log u_1)^\theta+(-\log u_2)^\theta)^{1/\theta}+\theta-1\right),$$

where $\theta\in[1,\infty)$.
The conditional copula is given by:

$$h_G(u_1,u_2,\theta_{12})=C_{12}(u_1,u_2;\theta_{12})\frac{1}{u_2}(-\log u_2)^{\theta_{12}-1}[(-\log u_1)^{\theta_{12}}+(-\log u_2)^{\theta_{12}}]^{1/\theta_{12}-1}.$$

There is not analytical formula for the inverse of the conditional copula, therefore it has to be be calculated numerically.
We use the rotated version of the Gumbel defined as: $C_{RG}(u_1,u_2,\theta)=u_1+u_2-1+C_G(1-u_1,1-u_2,\theta)$ and $c_{RG}(u_1,u_2,\theta)=c_G(1-u_1,1-u_2,\theta)$. $h_{RG}(u_1,u_2,\theta_{12})=1-h_G(1-u_1,1-u_2;\theta_{12})$. For the Rotated version of the Gumbel, $\lambda_L=2-2^{-1/\theta}$, $\lambda_U=0$.

7.2.4 Bivariate Clayton Copula

The Clayton copula has the following distribution

$$C_C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta},$$

and the following density:

$$c_C(u_1, u_2; \theta) = (1+\theta)(u_1 v_1)^{-\theta-1}(u_1^{-\theta} + u_2^{-\theta} - 1)^{-2-1/\theta},$$

where $\theta \in [-1, \infty)\backslash 0$.
The conditional copula is given by:

$$h_C(u_1, u_2, \theta_{12}) = u_2^{-1-\theta_{12}} \left(u_1^{-\theta_{12}} + u_2^{-\theta_{12}} - 1\right)^{-1-1/\theta_{12}}$$

and its inverse for a given value of the conditioning variable is:

$$h_C^{-1}(u_1, u_2, \theta_{12}) = \left((u_1 u_2^{\theta_{12}+1})^{-\theta_{12}/(\theta_{12}+1)} + 1 - u_2^{-\theta_{12}}\right)^{-1/\theta_{12}}$$

The Clayton copula has lower but not upper tail dependence: $\lambda_L = 2^{-1/\theta}$, $\lambda_U = 0$.

7.2.5 Bivariate Frank copula

The Frank copula has the following distribution

$$C_F(u_1, u_2; \theta) = -\frac{1}{\theta} log \left(\frac{\left(1 - e^{-\theta}\right) - \left(1 - e^{-\theta u_1}\right)\left(1 - e^{-\theta u_2}\right)}{\left(1 - e^{-\theta}\right)} \right),$$

and the following density:

$$c_F(u_1, u_2; \theta) = \frac{\theta\left(1 - e^{-\theta}\right) e^{-\theta(u_1+u_2)}}{\left(1 - e^{-\theta}\right) - \left(1 - e^{-\theta u_1}\right)\left(1 - e^{-\theta u_2}\right)},$$

where $\theta \in (-\infty, \infty)\backslash 0$.
The conditional copula is given by:

$$h_F(u_1, u_2, \theta_{12}) = \frac{(e^{-\theta_{12} u_1} - 1)(\theta_{12} e^{-\theta_{12} u_2})}{\theta_{12}((e^{-\theta} - 1) + (e^{-\theta_{12} u_1} - 1)(e^{-\theta_{12} u_2} - 1))},$$

and its inverse for a given value of the conditioning variable is:

$$h_F^{-1}(u_1, u_2, \theta_{12}) = -\log\left(\frac{e^{\theta_{12} u_2}(1 - u_1)/u_1 + e^{-\theta_{12}}}{1 + e^{-\theta_{12} u_2}(1 - u_1)/u_1} \right)/\theta_{12}.$$

The Frank copula has zero upper and lower tail dependence, $\lambda_U = \lambda_L = 0$, except in the limit when $\theta \to \infty$.

7.2.6 Mixture Copulas

In mixture models we can easily evaluate the tail dependence and Spearman rank correlation of the mixture, since they are linear combinations of their components:

$$\rho_S = p\rho_S^{(1)} + (1-p)\rho_S^{(2)}.$$

The same holds for tail dependence:

$$\lambda = p\lambda^{(1)} + (1-p)\lambda^{(2)}.$$

This is obvious from the definition. Denote $C(u,v) = \sum_{i=1}^{N} p_i C^{(i)}(u,v)$ a mixture copulas, with N components $C^{(i)}(u,v)$, each with weight p_i, $\sum_{i=1}^{N} p_i = 1$, and Spearman rank correlation ρ_S^i. Then the Spearman rank correlation ρ_S of the mixture copula is given by:

$$\rho_S = 12 \int\int_{[0,1]^2} C(u,v)dudv - 3 = \sum_{i=1}^{N} p_i \left(12 \int\int_{[0,1]^2} C^{(i)}(u,v)dudv - 3 \right) = \sum_{i=1}^{N} p_i \rho_S^{(i)}.$$

The same holds for tail dependence. Denote $\lambda^{(i)}$ the tail dependence of each component copula and λ the tail dependence of the mixture copula:

$$\lambda = \lim_{u \to 0^+} C(u,u)/u = \lim_{u \to 0^+} \sum_{i=1}^{N} p_i C^{(i)}(u,u)/u = \sum_{i=1}^{N} p_i \lambda^{(i)}.$$

The conditional copula for a mixture is a linear combination of the conditional copulas of their components. Denote $h^{(1)}$ and $h^{(2)}$ the conditional copulas for each component. Then, we have:

$$h_M = ph^{(1)} + (1-p)h^{(2)}.$$

The inverse of the conditional copula has to be evaluated numerically.

7.3 GARCH estimation results

	μ	AR1	AR2	Model	ω	α	γ	β	ν	λ
S&P 500	0.17	-0.11		GJR GARCH	0.14	0.00	0.17	0.88	14.51	-0.29
ENERGY	0.26			GJR GARCH	0.32	0.00	0.12	0.89	36.00	-0.20
XOM	0.28	-0.14		GARCH	0.94	0.10		0.78	8.71	-0.18
CVX	0.21			GARCH	4.31	0.17		0.31		
COP	0.25			GJR GARCH	0.76	0.01	0.10	0.87		
SLB	0.32			AVGARCH	0.09	0.05		0.94	12.66	
OXY	0.31			AVGARCH	0.12	0.05		0.93		
ESV	0.37			AVGARCH	0.10	0.06		0.94		
BJS	0.39			GARCH	1.13	0.11		0.87		
SUN	0.14			AVGARCH	0.12	0.05		0.93	6.47	
RDC	0.28			GARCH	0.32	0.05		0.94		
TSO	0.21			GJR GARCH	1.34	0.02	0.12	0.89	7.32	
INDUST	0.16			ZARCH	0.10	0.00	0.14	0.91	13.74	-0.19
GE	0.17			AVGARCH	0.05	0.07		0.93	8.18	
UTX	0.30			ZARCH	0.11	0.03	0.11	0.90	5.48	
BA	0.16			EGARCH	0.05	0.04	-0.07	0.98	8.16	
MMM	0.15			AVGARCH	0.03	0.04		0.96	6.02	
CAT	0.24			GJR GARCH	0.12	0.00	0.03	0.98	6.93	
PLL	0.11			GARCH	0.20	0.03		0.96	4.33	
AW	0.18			EGARCH	0.02	0.09	-0.04	0.99	5.68	
R	0.16			EGARCH	0.15	0.21	-0.07	0.95	10.24	
CTAS	0.11			GARCH	0.09	0.04		0.95	7.72	
RHI	0.26			ZARCH	0.52	0.04	0.14	0.83	5.35	
HEALTH	0.20	-0.11		AVGARCH	0.04	0.08		0.92	10.18	-0.16
JNJ	0.25	-0.15		GARCH	0.03	0.05		0.95	10.28	
PFE	0.15			GARCH	0.23	0.04		0.94	7.32	
MRK	0.10			AVGARCH	0.11	0.06		0.93	5.26	
ABT	0.18			GARCH	0.23	0.06		0.92	9.09	
WYE	0.15			GARCH	0.31	0.08		0.90	7.78	
MYL	0.07			GARCH	0.36	0.02		0.97	3.88	
MIL	0.18			GARCH	0.09	0.04		0.96	4.63	
PKI	0.19			GARCH	0.12	0.05		0.95	4.95	
WPI	0.12			AVGARCH	0.03	0.03		0.97	5.26	-0.12
THC	-0.08			GARCH	4.53	0.09		0.78	4.29	-0.13
FINAN	0.17	-0.14		EGARCH	0.08	0.23	-0.09	0.96	10.65	-0.23
BAC	0.12			EGARCH	0.06	0.21	-0.07	0.98	5.13	-0.16
JPM	0.16			GJR GARCH	0.34	0.02	0.16	0.89	8.38	-0.15
C	0.18			GARCH	0.12	0.09		0.91	5.79	
AIG	0.11	-0.14		ZARCH	0.25	0.03	0.18	0.84	9.73	
WFC	0.24	-0.13		EGARCH	0.02	0.17	-0.05	0.99	7.98	
HBAN	-0.07			GJR GARCH	0.17	0.04	0.11	0.90	4.73	
SOV	0.09	-0.12		EGARCH	0.07	0.16	-0.06	0.98	5.79	
MBI	-0.19			ZARCH	0.29	0.03	0.24	0.82	8.31	
FHN	-0.04			GARCH	0.14	0.08		0.91	6.74	
MTG	-0.10			ZARCH	0.19	0.02	0.15	0.90	5.65	
UTIL	0.10			AVGARCH	0.06	0.11		0.89	8.87	
EXC	0.28			AVGARCH	0.09	0.06		0.93	6.14	
SO	0.15			AVGARCH	0.08	0.12		0.88	6.78	
FPL	0.21	-0.10		AVGARCH	0.10	0.10		0.88	8.11	
D	0.14			AVGARCH	0.12	0.11		0.87	5.13	
DUK	0.07			AVGARCH	0.04	0.06		0.94	8.15	
TEG	0.11	-0.14		GARCH	0.10	0.06		0.93	8.62	
TE	0.01			GARCH	0.26	0.09		0.89	6.54	
PNW	0.07			GARCH	0.19	0.07		0.91	7.46	
CMS	-0.05	0.11	0.10	ZARCH	0.10	0.01	0.10	0.93	5.12	-0.13
GAS	0.09			EGARCH	0.08	0.08	-0.09	0.96	5.39	

MATERIALS	0.14			ZARCH	0.08	0.01	0.10	0.93	10.14	
DD	0.06			AVGARCH	0.03	0.04		0.96	11.29	
DOW	0.07			AVGARCH	0.05	0.06		0.94	8.20	
AA	0.18			GARCH	1.79	0.11		0.81	8.86	
PX	0.32			GJR GARCH	0.16	0.00	0.08	0.95	8.11	-0.16
NUE	0.24			GARCH	1.18	0.04		0.91	7.88	
SEE	0.12			ZARCH	0.14	0.00	0.14	0.92	3.79	
IFF	-0.02			GARCH	0.10	0.03		0.96	5.40	
ASH	0.10			EGARCH	0.02	-0.05	-0.03	0.99	6.11	-0.13
BMS	0.09			EGARCH	0.03	0.06	-0.08	0.99	6.16	
HPC	-0.11			EGARCH	0.06	0.06	-0.07	0.98	6.11	
C DISCR	0.12			EGARCH	0.07	0.13	-0.13	0.96		
MCD	0.20			GARCH	0.28	0.08		0.90	9.39	
CMCSA	0.23	-0.15		GARCH	0.25	0.07		0.92	7.59	
DIS	0.11			EGARCH	0.05	0.12	-0.10	0.98	7.48	
TWX	0.41			GARCH	0.08	0.07		0.93	12.35	
HD	0.13			EGARCH	0.08	0.17	-0.08	0.97	6.75	
KBH	0.15			GJR GARCH	1.43	0.00	0.11	0.91	6.71	
LIZ	0.08			GARCH	0.04	0.03		0.97	5.32	
JNY	0.12			EGARCH	0.05	0.15	-0.06	0.99	5.12	
MDP	0.12	0.11		ZARCH	0.12	0.07	0.11	0.87	6.22	
DDS	-0.10			AVGARCH	0.10	0.06		0.94	5.05	
C STAP	0.15			GARCH	0.04	0.07		0.92	9.47	
PG	0.20			AVGARCH	0.04	0.06		0.94	6.25	
WMT	0.27	-0.13		GARCH	0.08	0.04		0.96	7.59	
KO	0.11			GARCH	0.02	0.04		0.96	4.88	
PEP	0.23	-0.17		GARCH	0.05	0.07		0.93	9.26	
CVS	0.26			ZARCH	0.09	0.02	0.08	0.94	6.05	
BF.B	0.23			AVGARCH	0.13	0.10		0.88	10.83	
TSN	0.01			ZARCH	0.81	0.04	0.16	0.73	7.81	
WFMI	0.32			GARCH	0.13	0.02		0.98	4.03	
MKC	0.20			GARCH	0.42	0.05		0.90	5.47	
STZ	0.21			GARCH	0.00	0.01		0.99	3.85	
IT	0.19			GARCH	0.09	0.07		0.92		
MSFT	0.29			GARCH	0.08	0.05		0.95	4.12	0.14
IBM	0.27			AVGARCH	0.07	0.07		0.93	4.71	
AAPL	0.41			AVGARCH	0.06	0.03		0.97	4.72	
CSCO	0.36			GARCH	0.26	0.06		0.93	8.54	
INTC	0.25			GARCH	0.22	0.05		0.95	7.41	
TLAB	-0.04			AVGARCH	0.26	0.10		0.89	5.68	
NOVL	-0.14			AVGARCH	0.11	0.05		0.95	3.82	
CPWR	0.11			ZARCH	0.14	0.02	0.09	0.93	4.69	
QLGC	0.50			GARCH	0.04	0.06		0.94	5.34	
UIS	-0.11			GARCH	0.86	0.04		0.94	4.09	
TELECOM	0.05			GJR GARCH	0.23	0.01	0.10	0.91	8.72	
T	0.09	-0.10		AVGARCH	0.07	0.08		0.92	11.29	
VZ	0.06	-0.11		AVGARCH	0.08	0.07		0.92	6.50	
S	-0.01			ZARCH	0.18	0.03	0.12	0.90	5.19	
CTL	0.14			GARCH	0.13	0.05		0.95	4.51	
CZN	0.01			EGARCH	0.04	0.14	-0.07	0.99	4.93	

Table 10: Garch results

This table shows the results of the marginal models. Columns 2 to 4 contain the type of dynamics of the conditional mean. Column 5 to 12 contain the type of GARCH model specification that has been selected as well as the estimated parameters. ν is the degree of freedom for the Student-t/skewed Student t distribution and λ is the skewness parameter of the skewed Student t distribution.

7.4 Simulation Algorithm for VaR

In this section we explain the algorithm we use for simulating from the market-sector model. This algorithm generate one sample of random variables drawn from the CAVA model. Without loss of generality we assume that there are J sectors and in each sector there are I stocks. In the specific application of this paper $J = 10$. First we sample $(\omega_{rM}, \omega_{S_1}, \cdots, \omega_{S_J})$ independent uniform on $[0, 1]$ and then $(\omega_1^{S_1}, \omega_2^{S_1}, \cdots, \omega_I^{S_J})$ from a multivariate Gaussian copula with parameter Θ_G.

- for $j = 1$ to J
  - for $i = 1$ to I
    $\omega_i^{S_j} = h^{-1}_{S_j, r_i^{S_j} | r_M}(\omega_i^{S_j}, \omega_{S_j}; \theta_{S_j,i})$
    $\omega_i^{S_j} = h^{-1}_{M, r_i^{S_j}}(\omega_i^{S_j}, \omega_{r_M}; \theta^{S_j}_{M,i})$
    $r_i^{S_j} = F^{-1}(\omega_i^{S_j}; \alpha_i^{S_j})$
  - end for
    $\omega_{S_j} = h^{-1}_{M,S_j}(\omega_{S_j}, \omega_{r_M}; \theta_{M,S_j})$
    $r_{S_j} = F^{-1}(\omega_{S_j}; \alpha_{S_j})$
- end for
  $r_M = F^{-1}(\omega_{r_M}; \alpha_M)$

where $h^{-1}(\cdot)$ is the inverse of the conditional copula, see Section 7.2.