On the Non Gaussian Asymptotics of the Likelihood Ratio Test Statistic for Homogeneity of Covariance

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Abstract

The likelihood ratio test for \(m\)-sample homogeneity of covariance is notoriously sensitive to the violations of Gaussian assumptions. Its asymptotic behavior under non-Gaussian densities has been the subject of an abundant literature. In a recent paper, Yanagihara et al. (2005) show that the asymptotic distribution of the likelihood ratio test statistic, under arbitrary elliptical densities with finite fourth-order moments, is that of a linear combination of two mutually independent chi-square variables. Their proof is based on characteristic function methods, and only allows for convergence in distribution conclusions. Moreover, they require homokurticity among the \(m\) populations. Exploiting the findings of Hallin and Paindaveine (2008a), we reinforce that convergence-in-distribution result into a convergence-in-probability one—that is, we explicitly decompose the likelihood ratio test statistic into a linear combination of two variables which are asymptotically independent chi-square—and moreover extend it to the heterokurtic case.

1 Introduction.

Likelihood ratio tests (LRTs) for covariance matrices are notoriously sensitive to violations of strict Gaussian assumptions, and the problem of extending their validity to more general classes of distributions has generated a huge amount of literature. In a classical reference, Muirhead and Waternaux (1980) provide a general study of the problem of turning such tests into pseudo-Gaussian ones remaining valid under elliptical densities with adequate moment assumptions (typically, finite fourth-order moments). They clearly distinguish some “easy” cases—tests of sphericity, tests of equality of a subset of the characteristic roots of the covariance matrix (i.e., subspace sphericity), tests of block-diagonality—and some “harder” ones, among which the (apparently simpler) one-sample test of the hypothesis that the covariance matrix \(\Sigma\) takes some given value \(\Sigma_0\), the two-sample test of equality of covariance matrices, and the corresponding \(m\)-sample tests. The “easy cases” only require multiplying the traditional LRT statistic by some factor involving a consistent estimator of kurtosis; these cases have been fully characterized and solved by Shapiro and Browne (1987). As for the “hard” ones, Muirhead and Waternaux’s helpless conclusion is that “it is not possible in the more general elliptical case to adjust the (Gaussian likelihood ratio) test so that its limiting distribution agrees with that obtained under the normality assumption”.

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Muirhead and Waterman, as we shall see, were too pessimistic, and solutions since then have been obtained for some of the “hard” cases too—see Hallin and Paindaveine (2008b) for a general method. Nevertheless, it took another twenty years of uninterrupted efforts for such results (Schott (2001); Hallin and Paindaveine (2008a) and (2008c) for homogeneity of covariances) to appear, with contributions by Gupta and Tang (1984), Browne (1984), Hayakawa (1986), Shapiro and Browne (1987), Wakaki et al. (1990), Nagao and Srivastava (1992), Zhang and Boos (1992, 1993), Yuan and Bentler (1999), Tonda and Wakaki (2003), Yanagihara et al. (2005), Gupta and Xu (2006), Hallin and Paindaveine (2006), Paindaveine (2008), to name only a few.

Due to its fundamental role in multivariate analysis of variance and covariance, the problem of testing the null hypothesis $H_0$ of homogeneity of covariances, on which we concentrate here, is of special interest for applications. Denoting by $(X_{i1}, \ldots, X_{im})$, $i = 1, \ldots, m$ a collection of $m$ mutually independent samples of i.i.d. random $k$-dimensional vectors with location parameters $\theta_i$ and covariance matrices $\Sigma_i$, this null hypothesis writes $H_0 : \Sigma_1 = \ldots = \Sigma_m$. The Gaussian likelihood ratio test $\phi_{\text{LRT}}^{(n)}$ for this problem was first obtained by Wilks (1932), and rejects $H_0$ for large values of $Q^{(n)}_{\text{LRT}} := -2 \log \Lambda^{(n)}$, with

$$
\Lambda^{(n)} := \prod_{i=1}^{m} |W_i/n|^{n_i/2} =: \prod_{i=1}^{m} |S_i|^{n_i/2} / |S|^{n/2},
$$

(1.1)

where $\bar{X}_i := \frac{1}{n_i} \sum_{j=1}^{n_i} X_{ij}$, $W_i := \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)(X_{ij} - \bar{X}_i)' =: n_i S_i$, and $W := \sum_{i=1}^{m} W_i =: nS$. Even under Gaussian assumptions, this LRT is actually biased, and one therefore usually relies on the Bartlett (1937) modified LRT $\phi_{\text{MLRT}}^{(n)}$, based on $Q^{(n)}_{\text{MLRT}} := -2 \log \hat{\Lambda}^{(n)}$, where

$$
\hat{\Lambda}^{(n)} := \prod_{i=1}^{m} |W_i/\hat{n}_i|^{\hat{n}_i/2} =: \prod_{i=1}^{m} |\hat{S}_i|^{\hat{n}_i/2} / |S|^\hat{n}/2,
$$

(1.2)

with $\hat{n}_i := n_i - 1$ and $\hat{n} := \sum_{i=1}^{m} \hat{n}_i = n - m$; the statistics $Q^{(n)}_{\text{MLRT}}$ and $Q^{(n)}_{\text{LRT}}$, being asymptotically equivalent, however, we do not distinguish between the modified and the “unmodified” LRT anymore in the sequel, with a unique notation $Q_{\text{Wilks}}^{(n)}$.

## 2 A generalization of a result by Yanagihara et al. (2005).

In this context of testing covariance homogeneity, Yanagihara et al. (2005) show that, under homokurtic elliptical densities (when referring to homo- or heterokurticity, we of course tacitly assume the existence of finite fourth-order moments; the Yanagihara et al. result moreover holds under a slightly more general family of generalized elliptical densities), the asymptotic null distribution of $Q_{\text{Wilks}}^{(n)}$ is provided by

$$
Q_{\text{Wilks}}^{(n)} \xrightarrow{d} (1 + \kappa) \left\{ \left[ 1 + \frac{k\kappa}{2(1 + \kappa)} \right] Y_1 + Y_2 \right\},
$$

(2.1)

where $Y_1$ and $Y_2$ are independent chi-square random variables with $(m - 1)$ and $(m - 1)(k - 1)(k + 2)/2$ degrees of freedom, respectively, $\kappa$ stands for the common radial kurtosis of the $m$ underlying elliptical distributions, and $\xrightarrow{d}$ for convergence in distribution under the null. In the multinormal case, $\kappa = 0$, and (2.1) yields Wilks (1932)’s well-known Gaussian result that $Q_{\text{Wilks}}^{(n)}$ under the null hypothesis is asymptotically chi-square with $(m - 1)k(k + 1)/2$ degrees of freedom. But for $\kappa \neq 0$, (2.1) is no longer asymptotically chi-square (see also Gupta and Xu 2006).
The \((1 + \kappa)\) factor sitting in front of (2.1) is not uncommon in the context of likelihood ratio testing for covariance matrices (see Theorem 1 of Shapiro and Browne (1987) for a general result about this), and is very easily dealt with by dividing \(Q_{\text{Wilks}}^{(n)}\) by some consistent estimator \((1 + \hat{\kappa})\). The presence of \(\kappa\) in the coefficient of \(Y_1\), however, is more problematic. A corrected version of \(Q_{\text{Wilks}}^{(n)}\) indeed would require replacing \(Y_1\) and \(Y_2\) with \([1 + (k + 2)\hat{\kappa}/2]^{-1}Y_1\) and \((1 + \hat{\kappa})^{-1}Y_2\), respectively. The result in Yanagihara et al. (2005) being of a purely distributional nature, with proofs based on characteristic function methods, it does not provide any definition of \(Y_1\) and \(Y_2\) in terms of the observations, and such a correction therefore cannot be implemented.

Instead of (2.1), which is a convergence-in-distribution result, we provide here a convergence-in-probability result, with an explicit decomposition of \(Q_{\text{Wilks}}^{(n)}\) into a linear combination of explicitly defined and well-interpretable component variables. Our decomposition moreover also allows for heterokurtic populations.

As mentioned before, we throughout assume that all populations are elliptically symmetric and possess finite moments of order four. More precisely, defining, for proofs based on characteristic function methods, it does not provide any definition of \(Q\) for heterokurtic populations.

\[F^q := \{h : \mathbb{R}^+ \to \mathbb{R} : \mu_{k+q-1;h} < \infty\} \quad \text{and} \quad F^q_k := \{h \in F^q : \mu_{k+1;h} = k\},\]

respectively, where \(\mu_{\cdot;h} := \int_0^\infty r^h f(r) \, dr\), we require the following.

**Assumption (A).** The observations \(X_{ij}, j = 1, \ldots, n_i\) are mutually independent, with probability density function

\[x \mapsto c_{k,f_i|\Sigma_i}|^{-1/2}f_i \left(\left((x - \theta_i)/\Sigma_i^{-1}(x - \theta_i)\right)^{1/2}\right), \quad i = 1, \ldots, m, \quad (2.2)\]

for some \(k\)-dimensional vector \(\theta_i\) (location), some positive definite \((k \times k)\) covariance matrix \(\Sigma_i\), and some \(f_i\) in the class \(F^1_i\) of standardized radial densities with finite fourth-order moments.

Define (throughout, \(\Sigma^{1/2}\) stands for the symmetric root of \(\Sigma\)) the *elliptical coordinates*

\[U_{ij}(\theta_i, \Sigma_i) := \frac{\Sigma_i^{-1/2}(X_{ij} - \theta_i)}{||\Sigma_i^{-1/2}(X_{ij} - \theta_i)||} \quad \text{and} \quad d_{ij}(\theta_i, \Sigma_i) := ||\Sigma_i^{-1/2}(X_{ij} - \theta_i)||. \quad (2.3)\]

Under Assumption (A), the \(U_{ij}s, j = 1, \ldots, n_i, i = 1, \ldots, m\) are i.i.d. uniform over the unit sphere in \(\mathbb{R}^k\), and the *standardized elliptical distances* \(d_{ij}\) are independent of the \(U_{ij}\), with density \(f_{ik}(r) := (\mu_{k-1;f_i})^{-1/2}f_i(r)\) (justifying the terminology *standardized radial density* for \(f_i\)) and distribution function \(F_{ik}\). The condition that \(f_i \in F^1\) is equivalent to the finiteness of \(d_{ij}\’s\) fourth-order moments, while \(F^1_i \subset F^2_i\) implies that \(f_i\) is standardized in such a way that \(\text{E}[d_{ij}^2(\theta_i, \Sigma_i)] = k\), hence that \(\Sigma_i = \text{Var}[X_{ij}]\) is the covariance matrix in population \(i\).

Although, for the sake of notational simplicity, we do not mention it explicitly, we actually consider sequences of statistical experiments, with triangular arrays of observations of the form \(\{X_{11}^{(n)}, \ldots, X_{1n_i}^{(n)}; X_{21}^{(n)}, \ldots, X_{2n_i}^{(n)}; \ldots, X_{m1}^{(n)}, \ldots, X_{mn_m}^{(n)}\}\) indexed by the total sample size \(n\), where the sequences \(n_i^{(n)}\) of sample sizes satisfy the following assumption.

**Assumption (B).** For all \(i = 1, \ldots, m, n_i = n_i^{(n)} \to \infty\) as \(n \to \infty\).

Denoting by \(\kappa_i := [k(k + 2)]^{-1}\text{E}[d_{ij}^2(\theta_i, \Sigma_i)] - 1 = [k(k + 2)]^{-1}\int_0^1 (F_{ik}^{-1}(u))^4 \, du - 1\) the kurtosis coefficient in population \(i\) (see, e.g., page 54 of Anderson 2003), let, for all \(i \neq i'\),

\[\kappa_{i,i'} := \frac{n_i}{n_i + n_i'} \kappa_{i'} + \frac{n_i'}{n_i + n_i'} \kappa_i. \quad (2.4)\]

3
Proposition 1 Let Assumptions (A) and (B) hold. Then, under the null hypothesis $\mathcal{H}_0^{(n)}$ of covariance homogeneity,

$$Q^{(n)}_{\text{Wilks}} = \frac{1}{2kn} \sum_{1 \leq i<i' \leq m} n_i n_{i'} tr^2[S^{-1}(S_i - S_{i'})]$$

$$+ \frac{1}{2n} \sum_{1 \leq i<i' \leq m} n_i n_{i'} \left\{ tr[(S^{-1}(S_i - S_{i'}))^2] - \frac{1}{k} tr^2[S^{-1}(S_i - S_{i'})] \right\} + o_p(1)$$

$$= \frac{1}{n} \sum_{1 \leq i<i' \leq m} (n_i + n_{i'}) (1 + (k+2)\kappa_{i,i'}/2) Q^{(n)}_{\text{scale}}$$

$$+ \frac{1}{n} \sum_{1 \leq i<i' \leq m} (n_i + n_{i'}) (1 + \kappa_{i,i'}) Q^{(n)}_{\text{shape}} + o_p(1)$$

where the $m(m-1)/2$ statistics

$$Q^{(n)}_{\text{scale}} := \frac{n_i n_{i'}}{2k(n_i + n_{i'})} \frac{1}{1 + (k+2)\kappa_{i,i'}/2} tr^2[S^{-1}(S_i - S_{i'})]$$

are asymptotically chi-square, each with one degree of freedom (not mutually independent, though), in such a way that

$$Q^{(n)}_{\text{scale}} := \frac{1}{n} \sum_{1 \leq i<i' \leq m} (n_i + n_{i'}) Q^{(n)}_{\text{scale}}$$

is asymptotically chi-square with $(m-1)$ degrees of freedom, and the $m(m-1)/2$ statistics

$$Q^{(n)}_{\text{shape}} := \frac{n_i n_{i'}}{2(n_i + n_{i'})} \frac{1}{1 + \kappa_{i,i'}} \left\{ tr[(S^{-1}(S_i - S_{i'}))^2] - \frac{1}{k} tr^2[S^{-1}(S_i - S_{i'})] \right\}$$

are asymptotically chi-square, with $(k-1)(k+2)/2$ degrees of freedom each (but not mutually independent), in such a way that

$$Q^{(n)}_{\text{shape}} := \frac{1}{n} \sum_{1 \leq i<i' \leq m} (n_i + n_{i'}) Q^{(n)}_{\text{shape}}$$

is asymptotically chi-square with $(m-1)(k-1)(k+2)/2$ degrees of freedom; the $Q^{(n)}_{\text{scale}}$’s and $Q^{(n)}_{\text{shape}}$’s moreover are mutually independent for all $i, i', j, j'$; hence $Q^{(n)}_{\text{scale}}$ and $Q^{(n)}_{\text{shape}}$ also are mutually independent.

The proof of this proposition is provided in the Appendix.

An immediate consequence of this proposition is that, unless the $\kappa_{i,i'}$’s all vanish (which happens under Gaussian conditions), the asymptotic distribution of $Q^{(n)}_{\text{Wilks}}$ fails to be chi-square: Wilks’ test thus loses its validity under non-Gaussian densities—which includes, in particular, all heterokurtic situations.

The statistics $Q^{(n)}_{\text{scale}}$ and $Q^{(n)}_{\text{shape}}$ actually are the test statistics of the optimal pseudo-Gaussian tests obtained by Hallin and Paindaveine (2008a) for testing the subhypotheses $|\Sigma_i| = |\Sigma_{i'}|$ (equality of the generalized variances in populations $i$ and $i'$) and $\Sigma_i/|\Sigma_i|^{1/k} = \Sigma_{i'}/|\Sigma_{i'}|^{1/k}$ (equality of the shape matrices in populations $i$ and $i'$), respectively. The null hypothesis of covariance homogeneity clearly holds if and only if all those subhypotheses, each of which is associated with an “elementary contrast”, hold. The asymptotic equivalence (2.6) thus also
provides an interesting insight into the structure of Wilks’ statistic by decomposing $Q_{\text{Wilks}}(n)$ into a weighted sum of quadratic forms addressing those elementary contrasts. The way these contrasts are weighted, however, depends on the (unknown) kurtoses of the underlying populations; in particular, the relative importance given to scale and shape depends on the $\kappa_i$’s, which does not correspond to any sound decision-theoretical principle. The same reproach can be addressed to the bootstrapped versions of $Q_{\text{Wilks}}(n)$ developed in Zhang and Boos (1992, 1993).

It follows that the adequate correction, turning $Q_{\text{Wilks}}(n)$ into a pseudo-Gaussian test statistic with asymptotically chi-square distribution $((m-1)k(k+1)/2$ degrees of freedom) under arbitrary elliptical densities with finite fourth-order moments, is

$$Q_{\hat{Q}}(n) := \frac{1}{2n} \sum_{1 \leq i < i' \leq m} \frac{n_in_{i'}}{1 + (k + 2)\hat{\kappa}_{i,i'}} \frac{\text{tr}^2[S^{-1}(S_i - S_{i'})]}{2} - \frac{1}{k} \text{tr}^2[S^{-1}(S_i - S_{i'})],$$

(2.11)

where, denoting by $\hat{\kappa}_i, i = 1, \ldots, m,$ consistent estimators of the kurtosis coefficients $\kappa_i$,

$$\hat{\kappa}_{i,i'} := \hat{\kappa}_{i,i'} := \frac{n_i}{n_i + n_{i'}} \hat{\kappa}_{i'} + \frac{n_{i'}}{n_i + n_{i'}} \hat{\kappa}_i, \quad i = 1, \ldots, m,$$

have been substituted for the $\kappa_{i,i'}$’s in $Q_{\hat{Q}}(n)$ defined in (2.7) and (2.9), respectively. Contrary to the bootstrapping approach, this correction preserves $Q_{\text{Wilks}}(n)$’s nature of an unweighted sum of chi-squared variables, and does not affect its decision-theoretic properties. Note that $Q_{\hat{Q}}(n)$ is precisely the test statistic resulting from the general result of Hallin and Paindaveine (2008b) and proposed in Section 5.2 of Hallin and Paindaveine (2008a).

3 The homokurtic case.

In the homokurtic case—that is, when all kurtosis parameters $\kappa_i$ have some common value $\kappa$—definition (2.4) yields $\kappa_{i,i'}(n) = \kappa$ for all $n$, and (2.6) reduces to

$$Q_{\text{Wilks}}(n) = (1 + (k + 2)\kappa/2) \frac{1}{n} \sum_{1 \leq i < i' \leq m} (n_i + n_{i'}) Q_{\hat{Q}}(n),$$

$$+ (1 + \kappa) \frac{1}{n} \sum_{1 \leq i < i' \leq m} (n_i + n_{i'}) Q_{\hat{Q}}(n) + \text{op}(1)$$

$$= (1 + (k + 2)\kappa/2) Q_{\hat{Q}}(n) + (1 + \kappa) Q_{\hat{Q}}(n)$$

$$= (1 + \kappa) \left\{ \left[ 1 + \frac{k\kappa}{2(1 + \kappa)} \right] Q_{\hat{Q}}(n) + Q_{\hat{Q}}(n) \right\},$$

(3.1)
to be compared with (2.1). Denoting by $\hat{\kappa} = \hat{\kappa}(n)$ a consistent estimator of the common kurtosis $\kappa$, the corrected version of $Q_{\text{Wilks}}^{(n)}$ is thus

$$Q_{\text{Wilks}}^{(n)} = \frac{1}{2kn} (1 + (k + 2)\hat{\kappa}/2)^{-1} \sum_{1 \leq i < i' \leq m} n_in_{i'} \text{tr}[S^{-1}(S_i - S_{i'})^2]$$

$$+ \frac{1}{2n} (1 + \hat{\kappa})^{-1} \sum_{1 \leq i < i' \leq m} n_in_{i'} \{ \text{tr}[(S^{-1}(S_i - S_{i'}))^2] - \frac{1}{\hat{\kappa}} \text{tr}^2[S^{-1}(S_i - S_{i'})] \}, \quad (3.2)$$

which coincides with the homokurtic pseudo-Gaussian test statistic of Section 5.1 of Hallin and Paindaveine (2008a).

4 Appendix.

This appendix is devoted to the proof of Proposition 1. We first show that, under $H_0^{(n)}$,

$$Q_{\text{Wilks}}^{(n)} = \frac{1}{2n} \sum_{1 \leq i < i' \leq m} n_in_{i'} \text{tr}[(S^{-1}(S_i - S_{i'})^2] + o_p(1) \quad (4.1)$$

as $n \to \infty$. For notational convenience, we do not work with the Bartlett corrected version, but the result clearly remains valid for the latter. Letting $\Sigma^{1/2}Z_i\Sigma^{1/2} := n_i^{1/2}(S_i - \Sigma) =: n_i^{1/2}\Sigma Y_i$, where $\Sigma$ stands for the common null value of the $\Sigma_i$’s, and taking into account the fact that, as $||A|| \to 0$, $\log |I_k + A| = \text{tr} A - \frac{1}{2} \text{tr}(A^2) + o(||A||^2)$, we have that

$$Q_{\text{Wilks}}^{(n)} := -2\log \Lambda^{(n)} = -\sum_{i=1}^m n_i \log |S_i| + n \log |S|$$

$$= -\sum_{i=1}^m n_i \log |\Sigma(I_k + Y_i)| + n \log |\Sigma(I_k + (\frac{1}{n} \sum_{i=1}^m n_i Y_i))| + o_p(1)$$

$$= \frac{1}{2} \left\{ \sum_{i=1}^m n_i \text{tr}[Y_i^2] - \frac{1}{n} \text{tr} \left( \sum_{i=1}^m n_i Y_i \right)^2 \right\} + o_p(1)$$

$$= \frac{1}{2} \sum_{i=1}^m n_i \text{tr} \left( (Y_i - (\frac{1}{n} \sum_{i'=1}^m n_{i'} Y_{i'}))^2 \right) + o_p(1)$$

$$= \frac{1}{2} \sum_{i=1}^m n_i \text{tr} \left( \Sigma^{-1} (S_i - S)^2 \right) + o_p(1) = \frac{1}{2} \sum_{i=1}^m n_i \text{tr} \left[ S^{-1}(S_i - S)^2 \right] + o_p(1) \quad (4.2)$$

as $n \to \infty$, under the null hypothesis of covariance homogeneity and any elliptical density with finite fourth-order moments. Now,

$$\frac{1}{2} \sum_{i=1}^m n_i \text{tr} \left[ (S^{-1}(S_i - S))^2 \right] = \frac{1}{2} \sum_{i=1}^m n_i \text{tr} \left[ (S^{-1} \sum_{i'=1}^m \frac{n_{i'}}{n}(S_i - S_{i'})^2 \right]$$

$$= \frac{1}{2n^2} \sum_{i,i',i''=1}^m n_in_{i'}n_{i''} \text{tr} \left[ S^{-1}(S_i - S_{i'})S^{-1}(S_i - S_{i''}) \right].$$
Splitting $S_i - S_{i'}$ into $(S_i - S_{i'}) + (S_{i'} - S_{i''})$ then yields

$$
\frac{1}{2} \sum_{i=1}^{m} n_i \text{tr} \left[ (S^{-1}(S_i - S)) \right] = \frac{1}{2n} \sum_{i,i'=1}^{m} n_i n_{i'} \text{tr} \left[ (S^{-1}(S_i - S_{i'})) \right] - \frac{1}{2} \sum_{i=1}^{m} n_i \text{tr} \left[ (S^{-1}(S_i - S)) \right].
$$

Plugging this into (4.2), we obtain

$$
Q_{\text{Wilks}}^{(n)} = \frac{1}{4n} \sum_{i,i'=1}^{m} n_i n_{i'} \text{tr} \left[ (S^{-1}(S_i - S_{i'})) \right] + o_P(1)
$$

$$
= \frac{1}{2n} \sum_{1 \leq i < i' \leq m} n_i n_{i'} \text{tr} \left[ (S^{-1}(S_i - S_{i'})) \right] + o_P(1),
$$

which establishes (4.1), hence also (2.5). Inserting the statistics defined in (2.7) and (2.9), (2.5) takes the form (2.6). The asymptotic distributions of $Q_1^{(n)}$ scale and $Q_1^{(n)}$ shape in (2.8) and (2.10), and their relation to the subhypotheses of scale and shape homogeneity, respectively, have been derived in Section 6.1 of Hallin and Paindaveine (2008a). As for the asymptotic distributions of $Q_{1,i,i'}^{(n)}$ scale and $Q_{1,i,i'}^{(n)}$ shape, they follow from those of $Q_1^{(n)}$ scale and $Q_1^{(n)}$ shape by letting $m = 2.$

References


